Pairs of Generators for Matrix Groups. I

D. E. Taylor

School of Mathematics and Statistics
The University of Sydney
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It has been shown by Steinberg (1962) that every finite simple group of Lie type can be generated by two elements. These groups can be constructed from the simple Lie algebras over the complex numbers by the methods of Chevalley (1955), Steinberg (1959) and Ree (1961). The generators obtained by Steinberg (1962) are given in terms of the root structure of the corresponding Lie algebras.

The identification of the groups of Lie type A_n, B_n, C_n and D_n with classical matrix groups is due to Ree (1957). and an exposition of his results can be found in the book of Carter (1972). The proofs ultimately rely on the work of Dickson (1901). In this note we give tables of generators for the groups $GL(n,q)$, $SL(n,q)$, $Sp(2n,q)$, $U(n,q)$ and $SU(n,q)$. For the most part, the generators have been obtained by translating Steinberg's generators into matrix form via the methods of Ree (1957).

Notation

Let E_{ij} denote a square matrix with 1 in the (i,j)th position and 0 elsewhere. For $\alpha \in GF(q)$ and $i \neq j$ we set

$$x_{ij}(\alpha) = I + \alpha E_{ij}$$

and we let $h_i(\alpha)$ denote the diagonal matrix obtained by replacing the ith entry of the identity matrix by α. The $x_{ij}(\alpha)$ are the root elements of $SL(n,q)$.

Let w_i denote the monomial matrix obtained from the permutation matrix corresponding to the transposition $(i, i+1)$ by replacing the $(i+1, i)$-th entry by -1. Then $w = w_1 w_2 \ldots w_{n-1}$ represents the n-cycle $(1, 2, \ldots, n)$.

Let ξ be a generator of the multiplicative group of $GF(q)$.

1. $GL(n,q)$, $q \neq 2$

Generators for $GL(n,q)$ are

$$h_1(\xi) = \begin{pmatrix} \xi & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix} \quad \text{and} \quad x_{12}(1)w = \begin{pmatrix} -1 & 0 & \dots & 0 & 1 \\ -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}.$$

When $n = 2$, the generators are

$$\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Generators for $GL(n,2) = SL(n,2)$ are given below.

2. $SL(n,q)$, $q > 3$

The generators are

$$h_1(\xi)h_2(\xi^{-1}) = \begin{pmatrix} \xi & & & & \\ & \xi^{-1} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

and the matrix $x_{12}(1)w$ given above.

3. $SL(n,2)$ *and* $SL(n,3)$

The generators are

$$x_{12}(1) = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}.$$

4. *$Sp(2n, q)$, q odd, $n > 1$*

The symplectic group $Sp(2n, q)$ consists of the $2n \times 2n$ matrices X which satisfy the condition

$$X^t J X = J,$$

where

$$J = \left(\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{array}\right).$$

For $1 \leq i \leq n$, let $i' = 2n - i + 1$. Define

$$\hat{h}_i(\alpha) = h_i(\alpha) h_{i'}(\alpha^{-1})$$

and

$$\hat{x}_{ij}(\alpha) = x_{ij}(\alpha) x_{j'i'}(-\alpha).$$

Let $\hat{w}$ be the monomial matrix obtained from the permutation matrix of the $2n$-cycle

$$(1, 2, \ldots, n, 1', 2', \ldots, n')$$

by replacing the $(2n, n)$th entry by -1.

Generators for $Sp(2n, q)$, q odd, are

$$\hat{h}_1(\xi) = \left(\begin{array}{cccc|cccc} \xi & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & & & & & \xi^{-1} \end{array}\right)$$

and

$$\hat{x}_{12}(1)\hat{w} = \left(\begin{array}{ccccc|ccccc}
1 & 0 & & & & 1 & & & & \\
1 & 0 & & & & & & & & \\
0 & 1 & & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & 1 & 0 & 0 & & & & \\
\hline
 & & & 0 & 0 & 0 & 1 & & & \\
 & & & & & & 0 & 1 & & \\
 & & & & & & & \ddots & & \\
 & & & 0 & 1 & & & & 0 & 1 \\
 & & & 0 & -1 & & & & & 0
\end{array}\right).$$

When $n = 2$, the matrices are

$$\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Note that $Sp(2, q) \simeq SL(2, q)$.

5. *$Sp(2n, q)$, q even, $q \neq 2$, $n > 1$*

The $\hat{x}_{ij}(\alpha)$ are the short root elements of $Sp(2n, q)$. The long root elements are transvections $\hat{z}_i(\alpha) = x_{ii'}(\alpha)$.

Generators for $Sp(2n, q)$ are

$$\hat{h}_1(\xi)\hat{h}_n(\xi) = \left(\begin{array}{cccc|cccc}
\xi & & & & & & & \\
 & 1 & & & & & & \\
 & & \ddots & & & & & \\
 & & & \xi & & & & \\
\hline
 & & & & \xi^{-1} & & & \\
 & & & & & \ddots & & \\
 & & & & & & 1 & \\
 & & & & & & & \xi^{-1}
\end{array}\right)$$

and

$$\hat{x}_{1n}(1)\hat{z}_1(1)\hat{w} = \left(\begin{array}{ccccc|ccccc}
0 & 0 & & 1 & 1 & 1 & & & & \\
1 & 0 & & 0 & 0 & 0 & & & & \\
0 & 1 & & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & 1 & 0 & 0 & & & & \\
\hline
 & & & & 1 & 0 & 1 & & & \\
 & & & & 0 & & 0 & 1 & & \\
 & & & & & & & \ddots & & \\
 & & & & & & & & 0 & 1 \\
 & & & & 1 & & & & & 0
\end{array}\right).$$

For $Sp(4, q)$ the matrices become

$$\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & \xi^{-1} & 0 \\ 0 & 0 & 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

6. $Sp(2n, 2)$, $n > 2$

The generators are

$$\hat{x}_{1n}(1)\hat{z}_1(1) = \left(\begin{array}{ccc|ccc}
1 & & 1 & & & 1 \\
 & \ddots & & & & \\
 & & 1 & & & \\
\hline
 & & & 1 & & 1 \\
 & & & & \ddots & \\
 & & & & & 1
\end{array}\right)$$

and

$$\hat{w} = \left(\begin{array}{ccccc|ccccc}
0 & & & & & 1 & & & & \\
1 & 0 & & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & 0 & & & & & & \\
 & & & 1 & 0 & 0 & & & & \\
\hline
 & & & & 0 & 0 & 1 & & & \\
 & & & & & & 0 & & & \\
 & & & & & & & \ddots & & \\
 & & & & 0 & & & & 0 & 1 \\
 & & & & 1 & & & & & 0
\end{array}\right).$$

7. $Sp(4,2)$

The matrices are

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

8. $U(2n,q)$, $n > 1$

If $x \in GF(q^2)$, we set $\bar{x} = x^q$. If X is a matrix, $\overline{X}$ is obtained from X by replacing each entry x with $\bar{x}$. The unitary group consists of the matrices X such that

$$\overline{X}^t JX = J,$$

where

$$J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Let ξ be a primitive element of $GF(q^2)$ and let η be an element of trace 0, i.e., $\eta + \overline{\eta} = 0$. If q is odd, we may take $\eta = \xi^{(q+1)/2}$. If q is even, we may take $\eta = 1$.

For this section define $i' = 2n + 1 - i$ and set

$$\tilde{h}_i(\alpha) = h_i(\alpha)h_{i'}(\overline{\alpha}^{-1}),$$

and

$$\tilde{x}_{ij}(\alpha) = x_{ij}(\alpha)x_{j'i'}(-\overline{\alpha}),$$

for $1 \le i, j \le n$.

The matrix $\tilde{w}$ is similar to $\hat{w}$ of previous sections except that here it corresponds to the permutation $(1, 2, \ldots, n, 1', 2', \ldots, n')$ and it has η in the $(1, n+1)$th position and $-\eta^{-1}$ in the $(2n, n)$th position.

Generators for $U(2n,q)$ are

$$\tilde{h}_1(\xi)=\left(\begin{array}{cccc|cccc}
\xi &&&&&&& \\
& 1 &&&&&& \\
&& \ddots &&&&& \\
&&& 1 &&&& \\
\hline
&&&& 1 &&& \\
&&&&& \ddots && \\
&&&&&& 1 & \\
&&&&&&& \bar{\xi}^{-1}
\end{array}\right)$$

and

$$\tilde{x}_{12}(1)\tilde{w}=\left(\begin{array}{ccccc|cccccc}
1 & 0 &&&& \eta &&&&& \\
1 & 0 &&&&&&&&& \\
0 & 1 &&&&&&&&& \\
&& \ddots &&&&&&&& \\
&&& 1 & 0 & 0 &&&&& \\
\hline
&&&& 0 & 0 & 1 & 0 &&& \\
&&&&&& 0 & 1 &&& \\
&&&&&&&& \ddots && \\
&&&& \eta^{-1} &&&&& 0 & 1 \\
&&&& -\eta^{-1} &&&&&& 0
\end{array}\right).$$

When $n=2$ the matrices are

$$\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \bar{\xi}^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \eta & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \eta^{-1} & 0 & 1 \\ 0 & -\eta^{-1} & 0 & 0 \end{pmatrix}.$$

9. $SU(2n, q)$, $n > 1$

Generators are

$$\tilde{h}_1(\xi)\tilde{h}_2(\xi^{-1}) = \left(\begin{array}{cccc|cccc} \xi & & & & & & & \\ & \xi^{-1} & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & \bar{\xi} & \\ & & & & & & & \bar{\xi}^{-1} \end{array}\right)$$

and $\tilde{x}_{12}(1)\tilde{w}$.

10. $U(2n+1, q)$

In this section let $i' = 2n + 2 - i$ and define $\tilde{h}_i(\alpha)$ and $\tilde{x}_{ij}(\alpha)$ as before. In the following matrices the boxed entry is in position $(n + 1, n + 1)$.

For $\alpha, \beta \in GF(q^2)$ such that $\alpha\overline{\alpha} + \beta + \overline{\beta} = 0$, set

$$Q(\alpha, \beta) = \begin{pmatrix} I & & & & \\ & 1 & \alpha & \beta & \\ & & \boxed{1} & -\overline{\alpha} & \\ & & & 1 & \\ & & & & I \end{pmatrix}.$$

Let w' be the monomial matrix obtained from the permutation matrix of $(n', \ldots, 2', 1', n, \ldots, 2, 1)$ by replacing the $(n + 1, n + 1)$st entry by -1.

Let β be an element of $GF(q^2)$ such that $\beta + \overline{\beta} = -1$. We may take $\beta = -(1 + \bar{\xi}/\xi)^{-1}$.

Generators are

$$\tilde{h}_n(\xi) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \xi & & & & \\ & & & \boxed{1} & & & \\ & & & & \bar{\xi}^{-1} & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

and

$$Q(1,\beta)w' = \begin{pmatrix} 0 & 1 & & & & & & & & & \\ & 0 & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & 0 & 1 & & & & & & \\ \beta & & & & 0 & -1 & & & & & 1 \\ -1 & & & & & \boxed{-1} & & & & & \\ 1 & & & & & & 0 & & & & \\ & & & & & & 1 & 0 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & 0 & \\ & & & & & & & & & 1 & 0 \end{pmatrix}$$

11. $SU(2n+1,q)$, $n \neq 1$ *or* $q \neq 2$

Generators are

$$\tilde{h}_n(\xi)\tilde{h}_{n+1}(\xi^{-1}) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \xi & & & & \\ & & & \boxed{\bar{\xi}/\xi} & & & \\ & & & & \bar{\xi}^{-1} & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

and $Q(1,\beta)w'$ as above.

12. $SU(3,2)$

Generators are

$$\begin{pmatrix} 1 & \xi & \xi \\ 0 & 1 & \xi^2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \xi & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

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