On the dimension of affine domains

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Given a field k, an *affine domain over* k is an integral domain A which is finitely generated as a k-algebra; that is $A \simeq k[x_1, \dots, x_n]/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of a polynomial ring $k[x_1, \dots, x_n]$.

The *dimension* of A (denoted by $\dim A$) is the supremum of the integers n such that A contains a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals. The *height* of a prime ideal $\mathfrak{p}$ of A is the dimension of the localisation $A_\mathfrak{p}$ of A at $\mathfrak{p}$; that is, $\operatorname{ht}\mathfrak{p} = \dim A_\mathfrak{p}$.

Theorem 1. *Let $\mathfrak{p}$ be a prime ideal of an affine domain A and suppose that $\operatorname{ht}\mathfrak{p} = 1$. Then $\dim A/\mathfrak{p} = \dim A - 1$.*

The purpose of this note is to prove this theorem, based on the following results.

(i) A polynomial ring over a field is a unique factorisation domain.

(ii) Chapter 5 of [1] on (a) integral dependence, (b) the 'going-up' theorem, and (c) the 'going-down' theorem.

(iii) Noether's normalisation theorem as in [2, Theorem A.12].

The following lemma is a revised version of [2, Lemma A.13].

Lemma 2. *Let $A \subseteq B$ be integral domains, A integrally closed, B integral over A. Then $\dim A = \dim B$ and for all prime ideals $\mathfrak{q}$ of B, the ideal $\mathfrak{p} = \mathfrak{q} \cap A$ is a prime ideal of A such that $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q}$ and $\dim A/\mathfrak{p} = \dim B/\mathfrak{q}$.*

Proof. By the 'going-up' theorem, given a chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_s = \mathfrak{p} \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$$

of A, there is a chain of prime ideals $\mathfrak{q}_s \subset \cdots \subset \mathfrak{q}_n$ of B such that $\mathfrak{p}_i = \mathfrak{q}_i \cap A$ for $s \le i \le n$. And by the 'going-down' theorem there is a chain of prime ideals $\mathfrak{q}'_0 \subset \cdots \subset \mathfrak{q}'_s$ of B such that $\mathfrak{p}_i = \mathfrak{q}'_i \cap A$ for $0 \le i \le s$. Thus $\operatorname{ht}\mathfrak{p} \le \operatorname{ht}\mathfrak{q}$ and $\dim A/\mathfrak{p} \le \dim B/\mathfrak{q}$.

Conversely, if

$$\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_s = \mathfrak{q} \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$$

is a chain of prime ideals of B, then the ideals $\mathfrak{p}_i = \mathfrak{q}_i \cap A$ form a chain of distinct prime ideals of A. Hence $\operatorname{ht}\mathfrak{p} \ge \operatorname{ht}\mathfrak{q}$ and $\dim A/\mathfrak{p} \ge \dim B/\mathfrak{q}$. It follows that we have equalities $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q}$ and $\dim A/\mathfrak{p} = \dim B/\mathfrak{q}$. In particular, $\dim A = \dim B$. □

Lemma 3 (Nagata [3, Lemma (14.1)]). *Let $A = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. If f is an element of A which is not in k, then there exist algebraically independent elements y_1, y_2, ..., y_n in A with $y_1 = f$ such that A is integral over $k[y_1, \dots, y_n]$.*

Proof. Given n-tuples of integers $d = (d_1, \dots, d_n)$ and $e = (e_1, \dots, e_n)$ then $x^e = x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ is a monomial in A and we define its *weight* to be $d_1e_1 + \cdots + d_ne_n$. Choose d with $d_1 = 1$ so that no two monomials in f have the same weight and put $y_i = x_i - x_1^{d_i}$ for $2 \le i \le n$. Then $f = ax_1^h + g(x_1, y_2, \dots, y_n)$ where g is a polynomial whose degree in x_1 is less than h and where $a \in k$ is the coefficient of the term with the highest weight in f. Thus x_1 is integral over $B = k[f, y_2, \dots, y_n]$ and hence $x_i = y_i + x_1^{d_i}$ is integral over B for $2 \le i \le n$. Therefore A is integral over B and consequently the elements f, y_2, ..., y_n are algebraically independent. □

Lemma 4 (Nagata [3, Theorem (13.1)]). *If A is a unique factorisation domain then every prime ideal $\mathfrak{p}$ of height 1 in A is a principal ideal.*

Proof. Every nonzero element of A is a product of irreducible elements. Therefore, if $0 \ne r \in \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal, it follows that $\mathfrak{p}$ contains an irreducible factor f of r. In a unique factorisation domain, every irreducible element is prime and so the ideal (f) is prime. Thus if $\operatorname{ht} p = 1$, then $(f) = \mathfrak{p}$. □

Remark 5. Nagata also proved the converse of this result. Namely, if A is a Noetherian integral domain in which every prime ideal of height 1 is principal, then A is a unique factorisation domain. To prove this it suffices to show that every irreducible element is prime and this follows directly from Krull's principal ideal theorem.

Lemma 6. *If k is a field and $A = k[x_1, \dots, x_n]$, then* $\dim A = n$.

Proof. The chain of prime ideals $0 \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \dots, x_n)$ has length n and therefore $\dim A \ge n$. The proof proceeds by induction on n and the lemma is certainly true when $n = 0$. So suppose that $n > 0$ and let $0 = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ be a chain of prime ideals of A. We shall prove that $m \le n$.

Choose a non-zero element $r \in \mathfrak{p}_1$. Then r is a product of irreducible polynomials and since $\mathfrak{p}_1$ is prime, an irreducible factor of r belongs to $\mathfrak{p}_1$. Thus we may suppose that $\mathfrak{p}_1 = (f)$, for some f. By Lemma 3 there exist algebraically independent elements $y_1 = f$, y_2, ..., y_n in A such that A is integral over $B = k[y_1, \dots, y_n]$. Therefore $0 \subset \mathfrak{p}_1 \cap B \subset \cdots \subset \mathfrak{p}_m \cap B$ is a chain of prime ideals of B of length m and their images modulo $\mathfrak{q} = \mathfrak{p}_1 \cap B$ form a chain of prime ideals of length $m - 1$ in $B/\mathfrak{q} \simeq k[y_2, \dots, y_n]$. By induction $m - 1 \le n - 1$ and so $m \le n$, as required. □

Proof of Theorem 1. We begin with a prime ideal $\mathfrak{p}$ in an affine domain A over a field k such that $\operatorname{ht} \mathfrak{p} = 1$. By Noether's normalisation theorem there are algebraically independent elements x_1, ..., x_n in A such that A is integral over the polynomial ring $B = k[x_1, \dots, x_n]$.

By Lemma 2 the ideal $\mathfrak{q} = \mathfrak{p} \cap B$ is a prime ideal of B, $\mathrm{ht}(\mathfrak{q}) = 1$, $\dim A/\mathfrak{p} = \dim B/\mathfrak{q}$, and $\dim A = \dim B$. Thus we may replace A by B and assume that $A = k[x_1, \ldots, x_n]$.

It follows from Lemma 4 that there exists $f \in A$ such that $\mathfrak{p} = (f)$. By Lemma 3, there exist algebraically independent elements $y_1 = f$, $y_2, \ldots, y_n$ in A such that A is integral over $C = k[y_1, \ldots, y_n]$. By another application of Lemma 2 we may suppose that $A = C$. That is, we have reduced to the situation where $\mathfrak{p}$ is the prime ideal (y_1) in the polynomial ring $A = k[y_1, \ldots, y_n]$. Then $A/\mathfrak{p} \simeq k[y_2, \ldots, y_n]$ and by Lemma 6 we have $\dim A/\mathfrak{p} = n - 1 = \dim A - 1$, as required. □

References

[1] M. F. Atiyah and I. G. Macdonald. *Introduction to Commutative Algebra*. Addison-Wesley, Reading, Massachusetts, 1969.

[2] G. I. Lehrer and D. E. Taylor. *Unitary Reflection Groups*. Cambridge University Press, Cambridge 2009.

[3] M. Nagata. *Local rings*. Interscience Tracts in Pure and Applied Mathematics, No. 13. Interscience Publishers a division of John Wiley & Sons New York-London, 1962.