Convergence of bounded solutions of a degenerate parabolic problem on a bounded interval

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THE PROBLEM.

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Does for every continuous function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, which is Lipschitz continuous in the second variable, uniformly with respect to the first one, each bounded solution u of

$$(1) \qquad \begin{cases} u_t - \{|u_x|^{p-2}u_x\}_x + f(x,u) = 0 & \textit{in } (0,1) \times \mathbb{R}_+, \\ u(0,t) = u(1,t) = 0 & \textit{for } t \in \mathbb{R}_+, \end{cases}$$

converge to a solution φ of the stationary problem

$$(2) \qquad \begin{cases} -\{|\varphi_x|^{p-2}\varphi_x\}_x + f(x,\varphi) = 0 & \textit{in } (0,1), \\ \varphi(0) = \varphi(1) = 0. \end{cases}$$

as $t \to +\infty$?

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  - a parabolic maximum principle on non-cylindrical open sets,
  - the unique solvability of the initial value problem

$$-\varphi_{xx} + f(x, \varphi(x)) = 0 \quad \text{in } [0,1], \quad \varphi(x_0) = \varphi_0, \varphi_x(x_0) = \varphi_1$$

for given $x_0 \in [0,1]$, $\varphi_0, \varphi_1 \in \mathbb{R}$.

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- if u is a solution of problem (1), which is bounded with values in $L^2(0,1)$, then the set $\{u(t)\,|\,t \geq 1\}$ is relatively compact in $\mathrm{C}^1[0,1]$,
- a comparison principle for solutions of problem (1) on non-cylindrical open sets.

THE MAIN THEOREM.

THEOREM 1 (2011)

If $1 < p \leq 2$, then for every continuous function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, which is Lipschitz continuous in the second variable, w.r.t. the first one, each global solution of problem (1), which is bounded with values in $L^2(0,1)$, converges to a solution of the stationary problem (2) in $C^1[0,1]$ as $t \to +\infty$.

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For given $x_0 \in [0,1]$, $\varphi_0, \varphi_1 \in \mathbb{R}$ consider the initial value problem:

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$$\begin{pmatrix} \frac{d}{dx} v_1 \\ \frac{d}{dx} v_2 \end{pmatrix} = \begin{pmatrix} |v_2|^{\frac{2-p}{p-1}} v_2 \\ f(x, v_1) \end{pmatrix}, \quad v_1(x_0) = \tilde{\varphi}_0, v_2(x_0) = \tilde{\varphi}_1$$

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This IVP admits for every $x_0 \in [0,1]$, $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathbb{R}$ a unique solution provided $s \mapsto |s|^{\frac{2-p}{p-1}} s$ is locally Lipschitz on $\mathbb{R}$.

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$$s \mapsto |s|^{\frac{2-p}{p-1}} s \;\; \text{locally Lipschitz on } \mathbb{R} \qquad \text{iff} \qquad 1 < p \leq 2.$$

Existence of global solutions.

EXISTENCE OF GLOBAL SOLUTIONS.

Problem (1) can be rewritten as an abstract gradient system in $L^2(0,1)$ associated with the energy

$$\mathcal{E}(u) = \frac{1}{p}\int_0^1 |u_x|^p \, dx + \int_0^1 F(x, u(x)) \, dx, \quad \text{for all } u \in W_0^{1,p}(0,1).$$

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By the theory of subdifferential operators in Hilbert spaces (see Brézis [2, Lem. 6., Prop. 7., Prop. 8.]), for every $u_0 \in L^2(0,1)$, there exists a unique function

$$u \in \mathrm{C}(\mathbb{R}_+; L^2(0,1)) \cap W^{1,\infty}_{loc}((0,+\infty); L^2(0,1))$$

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... such that

- $u(\cdot, 0) = u_0(\cdot)$, and for all $t > 0$,

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- in every $t > 0$, u is differentiable from the right, and

$$\frac{du}{dt_+}(\cdot,t) - \{|u_x(\cdot,t)|^{p-2}u_x(\cdot,t)\}_x + f(\cdot,u(\cdot,t)) = 0 \quad \text{in } L^2(0,1),$$

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- $t \mapsto \mathcal{E}(u(t))$ is locally absolutely continuous on $(0, +\infty)$, and

$$\int_{t_1}^{t_2} \|u_t(t)\|^2_{L^2(0,1)}\, dt + \mathcal{E}(u(t_2)) = \mathcal{E}(u(t_1)) \qquad \text{for all } 0 < t_1 < t_2.$$

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  - Thus $(|u_x(\cdot,t_n)|^{p-2}u_x(\cdot,t_n))_{n\geq 1}$ is bounded in $W^{1,2}(0,1)$.

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$$\varphi^1_x(0) < \varphi^2_x(0) < \varphi^3_x(0).$$

- If $u_x(0,t) \geq \varphi^2_x(0)$ for all $t \geq t_0$, then

$$0 < \varphi^2_x(0) - \varphi^1_x(0) \leq \|u_x(\cdot,t) - \varphi^1_x\|_{C[0,1]}.$$

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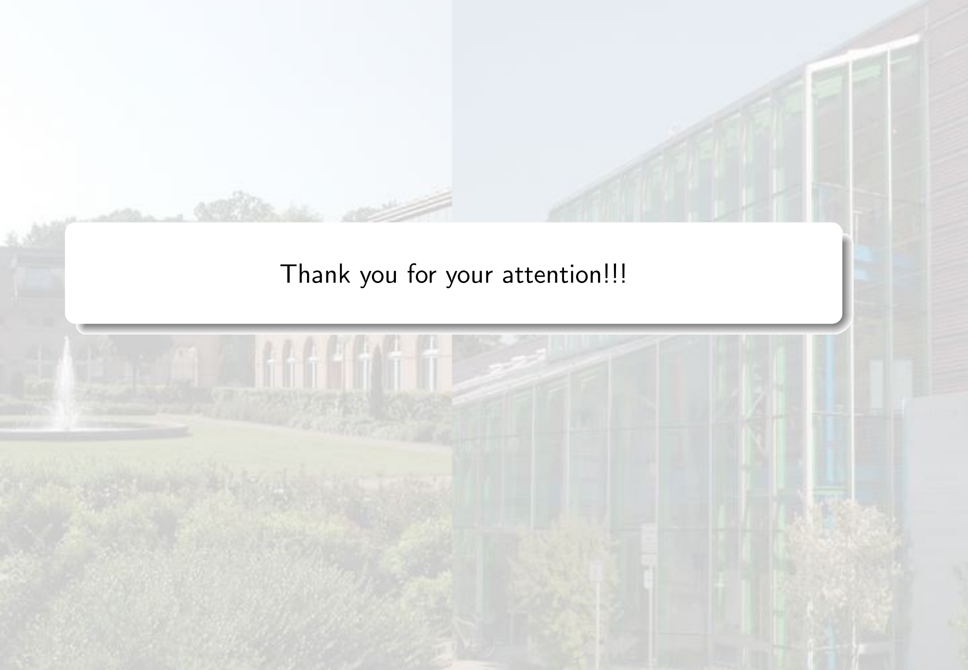

Thank you for your attention!!!

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- For $(x_0, t_0) \in \mathbb{R}^2$ and for $\rho > 0$, we set

$$\mathcal{Q}((x_0, t_0), \rho) := \Big\{ (x, t) \in \mathbb{R}^2 \;\Big|\; |x - x_0| < \rho,\; t_0 - \rho < t < t_0 \Big\}.$$

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- For any open subset $\mathcal{C} \subseteq \mathbb{R}^2$, we define by

$$\mathcal{P}\mathcal{C} := \big\{(x, t) \in \partial\mathcal{C} \,\big|\, \mathcal{Q}((x, t), \rho) \cap \mathcal{C}^c \neq \emptyset \text{ for all } \rho > 0\big\}$$

the *parabolic boundary* of $\mathcal{C}$,

THE PARABOLIC BDRY. OF A NON-CYLINDRICAL SET.

- For $(x_0, t_0) \in \mathbb{R}^2$ and for $\rho > 0$, we set

$$\mathcal{Q}((x_0, t_0), \rho) := \Big\{(x, t) \in \mathbb{R}^2 \,\Big|\, |x - x_0| < \rho,\ t_0 - \rho < t < t_0 \Big\}.$$

- For any open subset $\mathcal{C} \subseteq \mathbb{R}^2$, we define by

$$\mathcal{PC} := \big\{(x, t) \in \partial\mathcal{C} \,\big|\, \mathcal{Q}((x, t), \rho) \cap \mathcal{C}^c \neq \emptyset \text{ for all } \rho > 0\big\}$$

the *parabolic boundary* of $\mathcal{C}$,

- for every $T \in \mathbb{R}$, we define $\mathcal{C}_T := \big\{(x, t) \in \mathcal{C} \,\big|\, t < T\big\}$, and

THE PARABOLIC BDRY. OF A NON-CYLINDRICAL SET.

- For $(x_0, t_0) \in \mathbb{R}^2$ and for $\rho > 0$, we set

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  the *parabolic boundary* of $\mathcal{C}$,

- for every $T \in \mathbb{R}$, we define $\mathcal{C}_T := \{(x,t) \in \mathcal{C} \,|\, t < T\}$, and

- by t_{bot} the infimum of all $t \in \mathbb{R}$ for which there exists an $x \in \mathbb{R}$ such that $(x,t) \in \mathcal{C}$.

A COMPARISON PRINCIPLE.

LEMMA 1

Let $\mathcal{C} \subseteq \mathbb{R}^2$ be an open subset such that for all $T \in \mathbb{R}$, $\mathcal{C}_T$ is bounded and topological regular, that is, the interior $int(\overline{\mathcal{C}_T}) = \mathcal{C}_T$. If u and $v \in C(\overline{\mathcal{C}})$ satisfy for all bounded $(a_0, b_0) \times (t_0, t_1) \subseteq \mathcal{C}$,

$$u, v \in W^{1,2}(t_0, t_1; L^2(a_0, b_0)) \cap C([t_0, t_1]; W^{1,p}(a_0, b_0))$$

and for all non-negative $\xi \in C^1_c(\mathcal{C})$,

$$\int_{\mathcal{C}} [u_t - v_t]\, \xi \, d(x,t) + \int_{\mathcal{C}} \left[|u_x|^{p-2} u_x - |v_x|^{p-2} v_x \right] \xi_x \, d(x,t) + \int_{\mathcal{C}} [f(x,u) - f(x,v)]\, \xi \, d(x,t) \leq 0\,,$$

then

$$\sup_{(x,t) \in \overline{\mathcal{C}}} e^{-L(t-t_{bot})} (u-v)(x,t) \leq \sup_{(x,t) \in \mathcal{PC}} e^{-L(t-t_{bot})} \left[u - v\right]^+ (x,t).$$