2nd WOMASY - Meeting
Macquarie University 17. February, 2015

A simplified approach to the regularising effect of nonlinear semigroups

Daniel Hauer

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The motivation of this project:

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Since 2001 we observe that more and more papers appear on the topic:

"A L^q–L^r-regularising effect $(1 \le q < r \le \infty)$ of solutions of the equation

$$u_t - \Delta_p u^m = 0$$

depending on the regularity of the initial value $u(0)$ by taking advantage of a Log–Sobolev inequality"

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See, for instance:

- ▷ Cipriani & Grillo '00, '02
- ▷ Del Pino & Dolbeault '02, '04 + Gentil
- ▷ P. Takáč '04
- ▷ Bonforte & Grillo '05 ×2, '06
- ▷ Merkes '08, '09
- ▷ Warma '14

1st Lecture

"As soon as the underlaying space of weak solutions of a given elliptic equation

$$Au + f(u) = g(x)$$

satisfies a Sobolev-inequality, then each solution of this equation enjoys a L^q-L^r-regularisation effect.

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satisfies a Sobolev-inequality, then each solution of this equation enjoys a L^q-L^r-regularisation effect.

See, for instance:

- Gilbarg-Trudinger
- Daners '95

2nd Lecture

"Each (mild) solution of a given parabolic equation

$$\frac{du}{dt} + Au \ni 0$$

is the limit $\lim_{n\to\infty} u_n(t) = u(t)$ of solutions u_n of the elliptic equation

$$u_n + \frac{t}{n} A u_n \ni u_{n-1}\text{."}$$

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Remark

Of course, it would be more direct to use the regularising effect of the elliptic equation

$$u + \lambda A \ni f, \quad \lambda > 0.$$

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The Story

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Let $\{T_t\}_{t\geq 0}$ be a semigroup of bounded linear operators T_t acting on $L^q(\Sigma,\mu)$ for all $1\leq q\leq\infty$, of a measure space (Σ,μ), with infinitesimal gen. $-A$.

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In 1975, Gross considered the so-called "hypercontractivity":

for some (all) $1<q<r<\infty$, $\exists t:=t(q,r)>0$ s.th.

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$$T_t \text{ maps } L^q(\Sigma,\mu) \text{ to } L^r(\Sigma,\mu).$$

- Gross showed for a given measure space (Σ,μ) that $\{T_t\}_{t\geq 0}$ is hypercontractive iff Log-Sobolev holds for the generator $-A$ of $\{T_t\}_{t\geq 0}$.

The Story

▷ log-Sobolev inequality: Take $d\mu := (2\pi)^{-d/2} e^{-\frac{x^2}{2}} dx$

$$\int_{\mathbb{R}^d} |f|^2 \ln |f| \, d\mu \leq C \cdot \int_{\mathbb{R}^d} |\nabla f|^2 d\mu + \|f\|_2^2 \cdot \ln \|f\|_2$$

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▷ Here, it is important to know that hypercontractivity is a natural property of some infinite dimensional semigroups such as the Ornstein-Uhlenbeck semigroup.

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- In the 80's, in the context of heat kernels on Lie groups and manifolds, the focus shifted towards a stronger property, namely "ultracontractivity":

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  for all $t>0$, T_t maps $L^1(\Sigma,\mu)$ to $L^\infty(\Sigma,\mu)$.

- Of particular interest is the estimate

  (UE) $\quad \|T_t\|_{1\to\infty} \lesssim t^{-d/2}$ for every $t>0$, where

  $d>0$ plays the role of a dimension &

  $\|T_t\|_{q\to r} := \sup_{\|f\|_q\le 1} \|T_t f\|_r$ denotes the operator norm of $T_t \in \mathcal{L}(L^q(\Sigma,\mu), L^r(\Sigma,\mu))$.

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The Story

- In 1985, Varopoulos showed for a given (Σ,μ):

  $\{T_t\}_{t\geq 0}$ satisfies (UC) $\iff$ The generator $-A$ of $\{T_t\}_{t\geq 0}$ satisfies a d-dim Sobolev ineq.

  $$\|f\|^p_{\frac{pd}{d-2}} \leq C\cdot \langle Af, f_p\rangle$$

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by Varopoulos $\Longleftrightarrow$ The generator $-A$ of the O-U semigroup does not satisfy a d-dim. Sobolev ineq.

In other words, for the measure $d\mu = (2\pi)^{-d/2} e^{-x^2/2} dx$ a d-dim Sobolev inequality is not valid.

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In 1990, Davis (see his book about heat kernels) showed for a given (Σ, μ):

The semigroup $\{T_t\}_{t\geq 0}$ satisfies (UE) $\Longleftrightarrow$ there is a one parameter family of Log-Sobolev inequalities satisfied by the inf. generator $-A$.

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together with the fact that for $d\mu = dx$ Lebesgue measure

d-dim Sobolev $\Rightarrow$ Log-Sobolev inequality.

The Story

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This is the point where the nonlinear semigroup theory enters in the story!

L^q-L^r-regularising effect of nonlinear semigroups

- Bénilan's prototype operator was alway the porous media operator $\Delta\varphi$ for $\varphi(s)=|s|^{m-1}s$ but its theorems hold also for $\Delta_p\varphi$.

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- In 1979, Véron established the bounds (UC) for the semigroups generated by p-Laplace type operators and the porous media operator. He used first a Sobolev inequality and by using the notion of φ-accretivity, he could apply a simplified Moser-iteration to the semigroup.

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Need more regularity on the solutions.

Nonlinear semigroup theory

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- The set $D(A) := \{u \in X \mid Au \neq \emptyset\}$ is called domain of A.
- An operator A on X is called accretive in X if

$$\|u - \hat{u}\|_X \le \|u - \hat{u} + \lambda(v - \hat{v})\|_X \quad \text{for all } \lambda > 0 \text{ \& all } (u,v), (\hat{u},\hat{v}) \in A.$$

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$\iff$ the resolvent operator $J_\lambda := (1+\lambda A)^{-1}$ is a contraction.

Nonlinear semigroup theory

Let X be a real Banach space.

- We say that an accretive operator A on X is m-accretive in X if A satisfies the so-called range condition:

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- The condition "A is m-accretive" in X ensures that for all $u_0\in\overline{D(A)}^X$, the Cauchy problem

$$(CP)\quad \begin{cases} \frac{du}{dt}+Au\ni 0, & t>0\\ u(0)=u_0 \end{cases}$$

is well-posed in the sense of mild solutions.

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Nonlinear semigroup theory

Let X be a real Banach space.

▷ For given $u_0 \in X$, we call a function $u \in C([0,\infty);X)$ a strong solution of (EP) if $u \in W^{1,1}_{loc}((0,\infty);X)$, $u(0)=u_0$ in X and for a.e. $t>0$ one has

$$u(t) \in D(A) \quad \& \; -\frac{du}{dt}(t) \in Au(t).$$

Nonlinear semigroup theory

Let X be a real Banach space.

- Now, for given $u_0 \in X$ a function $u \in C([0,\infty);X)$ is called a mild solution of (EP) if for every $T>0$, $\varepsilon>0$ and partition $0=t_0<t_1<\dots<t_N=T$ of $[0,T]$ s.th. $t_i - t_{i-1} < \varepsilon$ $\forall i=1,\dots,N$ there is a piecewise constant function $u_{\varepsilon,N}(t) := \sum_{i=1}^{N} u_{\varepsilon,i} 1\!\!1_{(t_{i-1},t_i]}(t) + u_0$, where $u_{\varepsilon,i}$ on $(t_{i-1},t_i]$ solves "recursively" the finite difference equation $u_{\varepsilon,i} + (t_i - t_{i-1}) A u_{\varepsilon,i} \ni u_{\varepsilon,i-1}$ $\forall i=1,\dots,N$ such that $\sup_{t\in[0,T]} \|u_{\varepsilon,N}(t) - u(t)\|_X \leq \varepsilon$.

Nonlinear semigroup theory

Let X be a real Banach space.

▷ By the celebrated Crandall-Liggett Thm (1971), if A is m-accretive in X, then for every $u_0 \in \overline{D(A)}$ there is a unique mild solution of (EP) and the solution is given by the exponential formula

$$u(t) := \lim_{n\to\infty} \left(1+\frac{t}{n}A\right)^{-n} u_0 \quad \text{unif. on comp. subinterv. of } [0,\infty).$$

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▷ If we set $T_t u_0$ for every $u_0 \in \overline{D(A)}$ & $t \geq 0$,
then the family $\{T_t\}_{t\geq 0}$ defines a strongly continuous semigroup of contractions $T_t : \overline{D(A)} \to \overline{D(A)}$ w.r.t. X;

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more precisely,

(i) $T_{t+s} = T_t \circ T_s \quad \forall t \geq 0$ (semigroup property)

(ii) $\lim_{t \to 0+} \|u(t) - u_0\|_X = 0 \quad \forall u_0 \in \overline{D(A)}$ (strong cont.)

(iii) $\|T_t u - T_t \hat{u}\|_X \leq \|u - \hat{u}\|_X \quad \forall u, \hat{u} \in \overline{D(A)}, \forall t \geq 0$ (contract. prop.)

More regularity of nonlinear semigroups

- If the Banach space X has the property that

  X & its dual space X' are uniformly convex

  then for every $x_0 \in D(A)$, the mild solution $t \mapsto T_t x_0$ is a strong one.

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- If the Banach space X has the property that

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  then for every $u_0 \in D(A)$, the mild solution $t \mapsto T_t u_0$ is a strong one.

- If $X = H$ is a Hilbert space and $A = \partial\varphi$ the subgradient of a given convex, proper & l.s.c functional $\varphi : H \to (-\infty, \infty]$, then for every $u_0 \in \overline{D(\varphi)} = \overline{D(A)}$ the mild solution $t \mapsto T_t u_0$ is a strong one.

<u>The subgradient of a convex functional</u>

Let $X = H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$.

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▷ For a proper, convex function $\varphi : H \to (-\infty,\infty]$, the subgradient of φ is given by the set

$$\partial\varphi := \left\{(u,h)\in H\times H \,\middle|\, (h, v-u)_H \le \varphi(v)-\varphi(u) \ \forall v\in D(\varphi)\right\}$$

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Important prototype
$$\varphi(u) := \begin{cases} \int_{\mathbb{R}^d} |\nabla u|^p \, dx & \text{if } \nabla u \in L^p(\mathbb{R}^d)^d \\ +\infty & \text{if otherw.} \end{cases}$$
For every $u\in L^2(\mathbb{R}^d)$

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For every $u \in L^2(\mathbb{R}^d)$

$$\leadsto \partial\varphi = -\Delta_p^{\mathbb{R}^d}.$$

Different classes of operators

Let A be an operator on $M(\Sigma,\mu)$ of a given σ-finite measure space (Σ,μ).

The operators A, in which we are interested in can be classified into two groups:

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(II) A is T-accretive in $L^1(\Sigma,\mu)$ with non-increasing resolvent in $L^q(\Sigma,\mu)$ for all $1\le q\le\infty$.

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Ex.:
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Ex: $-\Delta u^m$ or $-\Delta_p u^m$.

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and every l.s.c., convex $j:\mathbb{R}\to[0,\infty]$ satisfying $j(0)=0$.

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and every l.s.c., convex $j:\mathbb{R}\to[0,\infty]$ satisfying $j(0)=0$.

By taking $j(s):=|s|^q$ for $1\le q<\infty$ & $j(s):=[s-k]^+$ for $k>0$ large enough we see that

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and every l.s.c., convex $j : \mathbb{R} \to [0,\infty]$ satisfying $j(0)=0$.

By taking $j(s) := |s|^q$ for $1 \le q < \infty$ & $j(s) := [s-k]^+$ for $k>0$ large enough we see that

$$A \text{ completely accretive} \implies A \text{ accretive in } L^q(\Sigma,\mu) \text{ for } 1 \le q \le \infty$$

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▷ By taking $j(s) := |[s]^+|^q$ for $1 \le q < \infty$, we see that

A completely accretive $\Longrightarrow$ A T-accretive in $L^q(\Sigma,\mu)$ for $1 \le q < \infty$.

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$$A \text{ completely accretive} \implies A \ T\text{-accretive in } L^{q}(\Sigma,\mu) \text{ for } 1 \le q < \infty.$$

▷ If $X \subseteq M(\Sigma,\mu)$ is a Banach lattice, then A is called T-accretive if

$$\|[u-\hat{u}]^{+}\|_X \le \|[u-\hat{u}+\lambda(v-\hat{v})]^{+}\|_X \quad \forall \lambda > 0 \ \& \ \forall (u,v), (\hat{u},\hat{v}) \in A.$$

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Notation

For $1 \leq q \leq \infty$ we set

$$A_q := \{(u,v) \in L^q(\Sigma,\mu) \times L^q(\Sigma,\mu) \mid (u,v) \in A\}$$

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- We say that A has a non-increasing resolvent if $\int j(u)\,d\mu \le \int j(u+\lambda v)\,d\mu$ for every $(u,v)\in A$ and every l.s.c. & convex $j:\mathbb{R}\to[0,\infty]$ with $j(0)=0$.

Sobolev ineq $\Rightarrow L^q$-L^r-bounds

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1. Theorem

Let A be completely accretive s.th. A_2 is m-accretive in $L^2(\Sigma,\mu)$ & $0 \in A0$ & A is densely defined.

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Suppose $\exists\, 1 < q < r \le \infty$ & $\exists\, c, \sigma > 0$ s.th. $\|u-\hat{u}\|_r^\sigma \le c \cdot \langle (u-\hat{u})_q, v-\hat{v} \rangle$

for every $(u,v), (\hat{u},\hat{v}) \in A_q$.

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Then the semigroup $\{T_t\}$ generated by $-A_2$ on $L^2(\Sigma,\mu)$ satisfies $\|T_t u-T_t\hat{u}\|_r \leq \left(\frac{c}{q}\right)^{1/\sigma}\cdot t^{-1/\sigma}\ \|u-\hat{u}\|_q^{q/\sigma}\quad \forall t>0$

and all $u,\hat{u}\in L^q(\Sigma,\mu)$.

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Sobolev ineq $\Rightarrow L^q$-L^r-bounds

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$$= q \int_0^t \left\langle A T_s u - A T_s \hat{u}, (u-\hat{u})_q \right\rangle ds$$

THE UNIVERSITY OF SYDNEY

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Sobolev ineq $\Rightarrow$ L^q-L^r-bounds

Proof of Theorem 1:

Let $u, \hat{u} \in D(A_q) \cap L^2(\Sigma, \mu)$.

$$\|u-\hat{u}\|_q^q \overset{\text{Hyp.}}{\geq} \frac{q}{C} \int_0^t \|T_s u - T_s \hat{u}\|_q^\sigma \, ds$$

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$$\geq \frac{q}{C} t \, \|T_t u - T_t \hat{u}\|_q^\sigma$$

Sobolev ineq $\Rightarrow L^q$-L^r-bounds

2. Theorem

Let A be an operator on $M(\Sigma,\mu)$ s.th. A_1 is m-accretive in $L^1(\Sigma,\mu)$ & has a non-increasing res.

Suppose $\exists\ 1 < q < r \leq \infty$ & $\exists\ c, \sigma > 0$ s.th. $\|u\|_q^\sigma \leq C \cdot \langle u_q, v\rangle$

for every $(u,v) \in A_q$ with $u \in D(A_1)$.

Then the semigroup $\{T_t\}$ generated by $-A_1$ on $L^1(\Sigma,\mu)$ satisfies
$$\|T_t u\|_r \leq \left(\frac{C}{q}\right)^{1/\sigma} \cdot t^{-1/\sigma}\, \|u\|_q^{q/\sigma} \qquad \forall t > 0$$
and all $u \in L^q(\Sigma,\mu) \cap \overline{D(A_1)}$

Sobolev ineq $\Rightarrow L^q$-L^r-bounds $\overset{\text{Moser}}{\Longrightarrow} L^q - L^\infty$-reg. effect

If $\exists\ \kappa > 1,\ 1 < p \le q_0$ s.th.

$$(\kappa-1)\,q_0 + (p-2) > 0$$

and if A_{q-p+2} satisfies

$$\|u-\hat{u}\|^q_{\kappa q} \lesssim \frac{(q/p)^p}{q-p+1} \langle v-\hat{v}, (u-\hat{u})_{q-p+2} \rangle$$

for all $(u,v), (\hat{u},\hat{v}) \in A_{q-p+2}$ with $u, \hat{u} \in L^\infty(\Sigma,\mu)$

then $\|T_t u - T_t \hat{u}\|_\infty \lesssim t^{-\delta} \|u-\hat{u}\|^\sigma_{\kappa q_0}$ $\forall\, u, \hat{u} \in L^{\kappa q_0}(\Sigma,\mu)$

with $\delta := \dfrac{1}{(\kappa-1)q_0 + p-2}$, $\sigma = \dfrac{(\kappa-1)q_0}{(\kappa-1)q_0+p-2}$

A nonlinear extrapolation result

The following result is a nonlinear generalisation of Lemma 1 in Coul '90 :

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Lemma

For $1 \le q < r \le \infty$, let $\{T_t\}$ be a L^1-contractive semigroup on $L^1(\Sigma,\mu) \cap L^r(\Sigma,\mu)$.

Suppose $\exists\, \alpha, \beta > 0$ & $C > 0$ s.th.

$$\|T_t u - T_t \hat{u}\|_r \le C\, t^{-\alpha} \|u - \hat{u}\|_q^{\beta} \quad \forall t > 0 \ \& \ \forall u, \hat{u} \in L^1 \cap L^r.$$

For $\theta_r := \frac{r-q}{q(r-1)} > 0$ if $r < \infty$ & $\theta_\infty := \frac{1}{q}$ if $r = \infty$

assume that $\beta(1 - \theta_r) < 1$.

Then $\|T_t u - T_t \hat{u}\|_r \le (2^{\alpha} C)^{1/\theta}\, t^{-\alpha/\theta} \|u - \hat{u}\|_1^{\gamma} \quad \forall t > 0 \ \& \ \forall u, \hat{u} \in L^1(\Sigma,\mu)$

where $\theta = 1 - \beta(1 - \theta_r) > 0$ & $\gamma := \beta \frac{\theta_r}{\theta}$.

Thank You!