An isoperimetric inequality related to a Bernoulli problem*

Daniel Daners Bernd Kawohl

28 January 2010

Dedicated to Giorgio Talenti on the occasion of his 70th birthday

Abstract

Given a bounded domain Ω we look at the minimal parameter $\Lambda(\Omega)$ for which a Bernoulli free boundary value problem for the p-Laplacian has a solution minimising an energy functional. We show that amongst all domains of equal volume $\Lambda(\Omega)$ is minimal for the ball. Moreover, we show that the inequality is sharp with essentially only the ball minimising $\Lambda(\Omega)$. This resolves a problem related to a question asked in [Flucher et al., J. Reine Angew. Math. **486** (1997), 165–204.].

Mathematics Subject Classification (2000). 35R35, 49R05
Keywords. isoperimetric inequality, Bernoulli problem, free boundary, symmetrisation

1 Motivation and Result

For given $\lambda > 0$ consider the following Bernoulli type free boundary problem

$$\begin{aligned} \Delta v &= 0 && \text{in } \Omega \setminus D, \\ v &= 0 && \text{on } \partial\Omega, \\ v &\equiv 1 && \text{on } D, \\ |\nabla v| &= \lambda && \text{on } \partial D, \end{aligned} \tag{1.1}$$

*This research is part of the ESF program "Global and geometric aspects of nonlinear partial differential equations (GLOBAL)".

The original publication is available at www.springerlink.com, Calculus of Variations and PDE's doi:10.1007/s00526-010-0324-4

on a given bounded open set $\Omega \subset \mathbb{R}^N$, where $D \subset \Omega$ is an unknown closed set. Such free boundary value problems originally arose from two dimensional flows (see [2, 7]), but also have applications to heat flows or electro-chemical machining (see the references in [4]).

It was shown in [1, Section 1.3] that some solutions to (1.1) can be obtained as non-trivial minimisers of the the functional

$$J_\lambda(u) := \int_\Omega |\nabla u(x)|^2 \, dx + \lambda^2 |\{u < 1\}| \tag{1.2}$$

over all $u \in H_0^1(\Omega)$ (Replace u by $1-u$ in [1]), where $\{u < 1\} := \{x \in \Omega \colon u(x) < 1\}$ and is $|\{u < 1\}|$ its Lebesgue measure. One can interpret the second term in J_λ as penalising the support of $(1-u)^+$. By reducing λ we expect the support of $(1-u)^+$ to grow or D to shrink. When we look at (1.1), we also expect $|\nabla u|$ to decrease as D shrinks. Hence the minimal λ for which a solution exists should occur when the distance between ∂D and $\partial\Omega$ becomes maximal. Therefore we expect an optimal configuration to maximise this distance, and a ball is very likely to do so. We set

$$\Lambda_2(\Omega) := \inf\{\lambda > 0 \colon J_\lambda \text{ has a non-trivial minimiser}\}.$$

and prove that $\Lambda_2(\Omega) \geq \Lambda_2(\Omega^*)$, where Ω^* denotes the ball of same volume as Ω. We also prove that equality holds if and only if Ω is a ball.

We will look at a more general problem. In [4] it is shown that for $1 < p < \infty$ non-trivial minimisers of the functional

$$J_{\lambda,p}(u) := \int_\Omega |\nabla u|^p \, dx + (p-1)\lambda^p |\{u < 1\}| \tag{1.3}$$

on $W_0^{1,p}(\Omega)$ solve the over-determined free boundary problem

$$\begin{aligned} \Delta_p v &= 0 \quad \text{in } \Omega \setminus D, \\ v &= 0 \quad \text{on } \partial\Omega, \\ v &\equiv 1 \quad \text{on } D, \\ |\nabla v| &= \lambda \quad \text{on } \partial D. \end{aligned} \tag{1.4}$$

Similarly as before we set

$$\Lambda_p(\Omega) := \inf\{\lambda > 0 \colon J_{\lambda,p} \text{ has a non-trivial minimiser}\}. \tag{1.5}$$

First we establish the following existence result.

Theorem 1.1. *The functional $J_{\lambda,p}$ has a non-trivial minimiser if and only if $\lambda \geq \Lambda_p(\Omega)$. Moreover,* $\min J_{\lambda,p} = J_{\lambda,p}(0)$ *if and only if $\lambda \leq \Lambda_p(\Omega)$.*

As zero is the only minimiser of $J_{0,p}(u) = \|\nabla u\|_p^p$ the above theorem implies that $\Lambda_p(\Omega) > 0$. Our main result is the following isoperimetric inequality. The proof of the sharpness of that inequality relies in an essential way on the fact from Theorem 1.1 that zero and a nontrivial $u \in W_0^{1,p}(\Omega)$ both minimize $J_{\Lambda_p(\Omega),p}$.

Theorem 1.2. *Let Ω be an arbitrary bounded domain in $\mathbb{R}^N$ and Ω^* a ball of same volume as Ω. Then*

$$\Lambda_p(\Omega) \geq \Lambda_p(\Omega^*), \tag{1.6}$$

with equality if and only if Ω is a ball up to a set of p-capacity zero. Moreover, if Ω^* has radius r then

$$\Lambda_p(\Omega^*) = \frac{p}{p-1}\left(\frac{p}{N}\right)^{(N-1)/(p-N)}\frac{1}{r}$$

if $N \neq p$ and

$$\Lambda_N(\Omega^*) = \frac{N}{N-1}e^{(1-1/N)}\frac{1}{r}.$$

if $N = p$.

Note that $\Lambda_p(\Omega^*)$ is a continuous function of $p \in (1, \infty)$. Also, if $p > N$, then points have positive p-capacity. Hence, if $\Lambda_p(\Omega) = \Lambda_p(\Omega^*)$ and $p > N$, then Ω is a ball.

Remark 1.3. If the integral $\int_\Omega |\nabla u|^p dx$ in $J_{\lambda,p}(u)$ is replaced by $\int_\Omega G(|\nabla u|)dx$, with suitable assumptions on G, including convexity of G, one can consider a more general quasi-linear equation for functions in the appropriate Orlicz space. Details of this can be found in [12].

A conjecture related to the above theorem appears in Flucher and Rumpf [5, page 202]. The difference is that we only look at solutions of (1.4) which minimise the energy functional $J_{\lambda,p}$, whereas [5] look at all solutions, that is,

$$\lambda_p(\Omega) := \inf\{\lambda > 0\colon (1.4) \text{ has a non-trivial solution}\}.$$

A comparison of the optimal constants on the ball as computed in Section 4 reveals that $\lambda_p(B) < \Lambda_p(B)$ if $\Omega = B$ is a ball. The new result in Theorem 1.1 is that there exists a non-trivial minimiser for $\lambda = \Lambda_p(\Omega)$. A similar result is proved in [9, Theorem 3.1] for $\lambda = \lambda_p(\Omega)$ and for convex Ω, but with completely different techniques to the ones we use. Also, [11] claim to prove the conjecture by Flucher and Rumpf but the proof seems flawed.

Since the energy minimising solutions have attracted quite some interest with the work in [1] in case $p = 2$ and [4] for general $p \in (1, \infty)$, our result should still be of interest. We give a proof of (1.6) in Section 3 and compute the optimal values in Section 4. Theorem 1.1 is proved in Section 2.

2 Existence of minimisers

In this section we establish the existence results for minimisers stated in Theorem 1.1. We throughout assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set.

Proposition 2.1. *Let $J_{\lambda,p}$ and $\Lambda_p(\Omega)$ be defined as in the previous section.*

(i) *If there exists $w \in W_0^{1,p}(\Omega)$ such that $J_{\lambda,p}(w) < J_{\lambda,p}(0) = (p-1)\lambda^p|\Omega|$, then $J_{\lambda,p}$ has a non-trivial minimiser.*

(ii) *For $\lambda > 0$ large enough $J_{\lambda,p}$ has a non-trivial minimiser.*

(iii) *Suppose $\mu > 0$ is such that $J_{\mu,p}$ has a nontrivial minimiser $u \in W_0^{1,p}(\Omega)$. Then $J_{\lambda,p}$ has a non-trivial minimiser for all $\lambda > \mu$.*

(iv) *We have $\min J_{\lambda,p} = J_{\lambda,p}(0)$ if and only if $\lambda \leq \Lambda_p(\Omega)$.*

Proof. (i) Since $J_{\lambda,p}(u) \geq 0$ for all $u \in W_0^{1,p}(\Omega)$ we can choose a minimising sequence $u_n \in W_0^{1,p}(\Omega)$ with $J_{\lambda,p}(u_n) \to \inf_{u \in W_0^{1,p}(\Omega)} J_{\lambda,p}(u)$. By definition of $J_{\lambda,p}$ the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$ and therefore has a subsequence (u_{n_k}) converging weakly in $W_0^{1,p}(\Omega)$ and pointwise almost everywhere in Ω to some function u. Hence

$$\|\nabla u\|_p^p \leq \liminf_{k\to\infty} \|\nabla u_{n_k}\|_p^p$$

and by Fatou's Lemma

$$\int_\Omega \chi_{\{u<1\}}\, dx \leq \liminf_{k\to\infty} \int_\Omega \chi_{\{u_{n_k}<1\}}\, dx,$$

where χ_A is the indicator function of a set $A \subseteq \mathbb{R}^N$ given by $\chi_A(x) = 1$ if $x \in A$ and zero otherwise. By definition of $J_{\lambda,p}$ and the choice of (u_n)

$$J_{\lambda,p}(u) \leq \liminf_{k\to\infty} J_{\lambda,p}(u_{n_k}) = \inf_{v \in W_0^{1,p}(\Omega)} J_{\mu,p}(v).$$

Thus, u is a minimiser. It is non-trivial since by assumption $J_{\lambda,p}(u) \leq J_{\lambda,p}(w) < J_{\lambda,p}(0)$.

(ii) Let $\varphi \in C_c^\infty(\Omega)$ such that $|\{\varphi \geq 1\}| > 0$. Then note that

$$J_{\lambda,p}(\varphi) - J_{\lambda,p}(0) = \|\nabla\varphi\|_p^p - (p-1)\lambda^p|\{\varphi \geq 1\}| < 0$$

for $\lambda > 0$ large enough. Now apply (i).

(iii) Clearly, if u is a non-trivial minimiser of $J_{\mu,p}$, then $J_{\lambda,p}(u) \leq J_{\lambda,p}(0)$. Also, $|\{u < 1\}| < |\Omega|$ since otherwise $J_{\lambda,p}(0) < J_{\lambda,p}(u)$ and u is not a minimiser. Hence from the definition of $J_{\mu,p}$ we have

$$\begin{aligned} J_{\lambda,p}(u) &= \|\nabla u\|_p^p + (p-1)\lambda^p|\{u<1\}| \\ &= J_{\mu,p}(u) + (p-1)(\lambda^p - \mu^p)|\{u<1\}| \\ &\leq (p-1)\mu^p|\Omega| + (p-1)(\lambda^p - \mu^p)|\{u<1\}| \\ &= (p-1)\lambda^p|\Omega| - (p-1)(\lambda^p-\mu^p)\big(|\Omega| - |\{u<1\}|\big). \end{aligned} \tag{2.1}$$

Since $|\{u < 1\}| < |\Omega|$ we conclude that $J_{\lambda,p}(u) < (p-1)\lambda^p|\Omega| = J_{\lambda,p}(0)$ for all $\lambda > \mu$. By (i) $J_{\lambda,p}$ has a non-trivial minimiser for all $\lambda > \mu$.

(iv) If $\lambda < \Lambda_p(\Omega)$, then clearly $\min_{u\in W_0^{1,p}(\Omega)} J_{\lambda,p}(u) = J_{\lambda,p}(0)$, so assume that $\mu := \Lambda_p(\Omega)$. Assume that u is a minimiser of $J_{\mu,p}$ and suppose that strict inequality holds in (2.1). Then clearly $J_{\lambda,p}(u) < J_{\lambda,p}(0) = (p-1)\lambda^p|\Omega|$ if $\lambda < \mu$ is close enough to μ. However, this contradicts the definition of $\mu = \Lambda_p(\Omega)$ since otherwise (i) implies the existence of a minimiser for some $\lambda < \mu$. □

To prove that $J_{\lambda,p}$ also has a non-trivial minimiser for $\lambda = \Lambda_p(\Omega)$ we need to compare $\|\nabla u\|_p$ with the measure of $\{u \geq 1\}$. In the following lemma we get such an estimate. It is motivated by the estimate of the measure of a set in terms of its capacity (see e.g. [6, page 5]), but does not rely on capacity.

Lemma 2.2. *Let $1 < p \leq N$. Then there exist $q > p$ and $C > 0$ only depending on N, p and $|\Omega|$ such that $|\{u \geq 1\}| \leq C\|\nabla u\|_p^q$ for all $u \in W_0^{1,p}(\Omega)$.*

Proof. If $1 < p < N$, by the Sobolev inequality there exists a constant $C > 0$ only depending on N and p such that

$$|\{u \geq 1\}| \leq \int_\Omega |u|^{Np/(N-p)}\,dx \leq C\|\nabla u\|_p^{Np/(N-p)}$$

for all $u \in W_0^{1,p}(\Omega)$. Hence we can set $q := Np(N-p) > p$. If $p = N$ choose $p_0 \in (N/2, N)$ and apply the above inequality and Hölder's inequality to get

$$|\{u \geq 1\}| \leq C\|\nabla u\|_{p_0}^{Np_0/(N-p_0)} \leq C|\Omega|^\theta\|\nabla u\|_p^{Np_0/(N-p_0)}$$

for all $u \in W_0^{1,p}(\Omega)$, where θ is a constant depending only on p_0 and N. Hence we can set $q := Np_0/(N-p_0)$. Clearly $q > N$ since $p_0 > N/2$. □

Since by definition of $\Lambda_p(\Omega)$ the functional $J_{\lambda,p}$ has no non-trivial minimiser for $\lambda < \Lambda_p(\Omega)$ the following proposition will conclude the proof of Theorem 1.1. It is also the most original and new part of the proof.

Proposition 2.3. *If $\mu = \Lambda_p(\Omega)$, then $J_{\mu,p}$ has a non-trivial minimiser.*

Proof. By definition of $\Lambda_p(\Omega)$ either there exists a non-trivial minimiser or there is a sequence (λ_n) such that $\lambda_n > \mu$ for all $n \in \mathbb{N}$, $\lambda_n \to \mu$ and $J_{\lambda_n,p}$ has a non-trivial minimiser $u_n \in W_0^{1,p}(\Omega)$ for every $n \in \mathbb{N}$. Then

$$J_{\lambda,p}(u_n) = \|\nabla u_n\|_p^p + (p-1)\lambda_n^p|\{u_n < 1\}| \leq (p-1)\lambda_n^p|\Omega|$$

for all $n \in \mathbb{N}$. Since (λ_n) is a convergent sequence, (u_n) is bounded in $W_0^{1,p}(\Omega)$. It therefore has a convergent subsequence such that $u_{n_k} \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and pointwise almost everywhere. Fix $v \in W_0^{1,p}(\Omega)$. As in the proof of Proposition 2.1(i)

$$J_{\mu,p}(u) \leq \liminf_{k\to\infty} J_{\lambda_{n_k},p}(u_{n_k}) \leq \liminf_{k\to\infty} J_{\lambda_{n_k},p}(v) = J_{\mu,p}(v) \tag{2.2}$$

where in the second inequality we use that u_{n_k} are minimisers for $J_{\lambda_{n_k},p}$. Hence $u \in W_0^{1,p}(\Omega)$ is a minimiser of $J_{\mu,p}$.

To conclude the proof we need to show that $u \neq 0$. If $u = 0$ and $p > N$, then $u_{n_k} \to 0$ uniformly as $k \to \infty$ since $W_0^{1,p}(\Omega)$ is compactly embedded into $C(\bar{\Omega})$. Therefore there exists $m \in \mathbb{N}$ such that $\|u_m\|_\infty < 1$ and so

$$J_{\lambda_m,p}(u_m) = \|\nabla u_m\|_p^p + (p-1)\lambda_m^p|\Omega| > (p-1)\lambda_m^p|\Omega| = J_{\lambda_m,p}(0)$$

since by assumption $u_m \neq 0$. As u_m was a non-trivial minimiser this is a contradiction, and so $u \neq 0$.

We next look at the case $1 < p \leq N$. Again assume that $u = 0$. Then by Rellich's theorem we have $u_{n_k} \to 0$ in $L_p(\Omega)$ and so

$$|\{u_{n_k} \geq 1\}| \leq \int_{\{u_{n_k}\geq 1\}} |u_{n_k}|^p\,dx \leq \|u_{n_k}\|_p^p \to 0$$

Hence (2.2) with $u = v = 0$ implies that

$$\begin{aligned}\mu^p(p-1)|\Omega| = J_{\mu,p}(0) &\leq \liminf_{k\to\infty} J_{\lambda_{n_k},p}(u_{n_k})\\ &\leq \liminf_{k\to\infty} J_{\lambda_{n_k},p}(0) = J_{\mu,p}(0) = \mu^p(p-1)|\Omega|,\end{aligned}$$

and therefore $\|\nabla u_{n_k}\|_p \to 0$. As $J_{\lambda_n,p}(0) = (p-1)\lambda_n^p|\Omega|$, Lemma 2.2 implies the existence of constants $C > 0$ and $q > p$ such that

$$\begin{aligned}J_{\lambda_n,p}(u_n) &= J_{\lambda_n,p}(0) + \|\nabla u_n\|_p^p - (p-1)\lambda_n^p|\{u_n \geq 1\}|\\ &\geq J_{\lambda_n,p}(0) + \|\nabla u_n\|_p^p - C(p-1)\lambda_n^p\|\nabla u_n\|_p^q\\ &= J_{\lambda_n,p}(0) + \|\nabla u_n\|_p^p\Big(1 - C(p-1)\lambda_n^p\|\nabla u_n\|_p^{q-p}\Big)\end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$. Since $\|\nabla u_{n_k}\|_p \to 0$, $\lambda_n \to \mu$ and $q > p$ there exists $m \in \mathbb{N}$ with

$$1 - C(p-1)\lambda_m^p \|\nabla u_m\|_p^{q-p} > 0$$

and hence by (2.3) we get $J_{\lambda_m,p}(u_m) > J_{\lambda_m,p}(0)$. This is a contradiction since u_m was assumed to be a minimiser for $J_{\lambda_m,p}$, and so $u \neq 0$ as claimed. □

3 Proof of the isoperimetric inequality

This whole section is devoted to the proof of the first part of Theorem 1.2. We let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $\Omega^* \subset \mathbb{R}^N$ be an open ball with the same volume as Ω. For $v \in W_0^{1,p}(\Omega^*)$ set

$$J_{\lambda,p}^*(v) = \int_{\Omega^*} \big(|\nabla v|^p \, dx + (p-1)\lambda^p |\{v < 1\}|$$

and recall that minimisers are solutions of (1.4) with Ω replaced by Ω^*. Let $\lambda \geq \Lambda_p(\Omega)$. By Theorem 1.1 $J_{\lambda,p}$ has a non-trivial minimiser $u \in W_0^{1,p}(\Omega)$. Consider its Schwarz symmetrisation u^* (see [10, 13] for a definition and properties). By well known properties of Schwarz symmetrisation $u^* \in W_0^{1,p}(\Omega^*)$, $\|\nabla u^*\|_p \leq \|\nabla u\|_p$ and $|\{u^* < 1\}| = |\{u < 1\}|$. Also u^* is non-zero and

$$\begin{aligned} J_{\lambda,p}^*(u^*) &= \|\nabla u^*\|_p^p + (p-1)\lambda^p |\{u^* < 1\}| \\ &\leq \|\nabla u\|_p^p + (p-1)\lambda^p |\{u < 1\}| = J_{\lambda,p}(u). \end{aligned} \tag{3.1}$$

In particular $J_{\lambda,p}^*(u^*) \leq J_{\lambda,p}(u) \leq J_{\lambda,p}(0) = J_{\lambda,p}^*(0)$. If $J_{\lambda,p}^*(u^*) < J_{\lambda,p}^*(0)$, then by Proposition 2.1(i) $J_{\lambda,p}^*$ has a non-trivial minimiser. If $J_{\lambda,p}^*(u^*) = J_{\lambda,p}^*(0)$, then either u^* is a non-trivial minimiser, or $\inf J_{\lambda,p}^* < (p-1)\lambda^p|\Omega^*|$ and Proposition 2.1(i) implies the existence of a non-trivial minimiser. In any case, if $J_{\lambda,p}$ has a non-trivial minimiser, so does $J_{\lambda,p}^*$. Hence by definition of $\Lambda_p(\Omega)$ and $\Lambda_p(\Omega^*)$ the inequality (1.6) follows.

It remains to prove the sharpness of (1.6). We assume that $\Lambda_p(\Omega) = \Lambda_p(\Omega^*)$. The aim is to show that Ω is a ball up to a set of capacity zero. To simplify notation we denote the common value of $\Lambda_p(\Omega)$ and $\Lambda_p(\Omega^*)$ by Λ and let r be the radius of the ball Ω^*. By Theorem 1.1 zero is a minimiser for the problem on Ω and also on Ω^*. Hence, using (3.1)

$$(p-1)\Lambda^p|\Omega| = J_{\Lambda,p}^*(0) \leq J_{\Lambda,p}^*(u^*) \leq J_{\Lambda,p}(u) = J_{\Lambda,p}(0) = (p-1)\Lambda^p|\Omega|.$$

We conclude that $J_{\Lambda,p}^*(u^*) = J_{\Lambda,p}(u)$. In particular, u^* is a minimiser of $J_{\Lambda,p}^*$. Since there is a unique radially symmetric minimiser on Ω^* (see the argument at the start of Section 4) we conclude that u^* coincides with (4.1) if $p \neq N$ and (4.2) if $p = N$ with ρ given by (4.6) and (4.8), respectively. In particular, $\nabla u^*(x) = \nabla u_\rho(x) \neq 0$ whenever $0 < u_\rho(x) < 1 = \max u_\rho$. Therefore, [3, Theorem 1.1] applies and so, up to translation, $u = u^* = u_\rho$ almost everywhere. Extending u, u^* by zero outside Ω and Ω^*, respectively we can assume that $u, u^* \in W^{1,p}(\mathbb{R}^N)$. We can then replace u and u^* by a quasi-continuous representative as defined in [8, Theorem 4.5]. Since u_ρ is continuous and $u^* = u_\rho$ almost everywhere, u_ρ is the quasi-continuous representative of u^*. Hence $u_\rho = u$ quasi everywhere, that is, except possibly on a set of p-capacity zero. Also, as $u \in W_0^{1,p}(\Omega)$ we know from [8, Theorem 4.5] that $u = 0$ quasi everywhere on Ω^c. Combining the two facts we get $u = u_\rho = 0$ quasi-everywhere on $C := \Omega^* \setminus \Omega$. Since $u_\rho > 0$ on Ω^* we conclude that C must have p-capacity zero. Hence $\Omega = \Omega^*$ is a ball except possibly for a set of p-capacity zero.

4 The optimal constants

In this section we look at (1.4) in case $\Omega = B_r$ is a ball of radius $r > 0$ centred at the origin. We want to compute the value of $\Lambda_p(B_r)$. To do so we assume that $\lambda \geq \Lambda_p(B_r)$ and that $u \in W_0^{1,p}(B_r)$ is a minimiser of $J_{\lambda,p}$. Let $u^* \in W_0^{1,p}(B_r)$ be its Schwarz symmetrisation. According to (3.1) we have $J_{\lambda,p}(u^*) \leq J_{\lambda,p}(u)$. Hence there is a radially symmetric minimiser u_ρ and we can assume without loss of generality that $u_\rho = u_\rho^*$. Let $\rho > 0$ be the radius of the ball $\{u \geq 1\}$. By [4, Theorem 2.1] (or [1, Lemma 2.4] in case $p = 2$) the minimiser is p-harmonic on $B_r \setminus \bar{B}_\rho$ with $u = 0$ on ∂B_r and $u = 1$ on ∂B_ρ. As there is precisely one such p-harmonic function (see [8, Lemma 8.5])

$$u_\rho(x) = \begin{cases} \dfrac{|x|^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}}{\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}} & \text{if } \rho \leq |x| \leq r \\ 1 & \text{if } 0 \leq |x| \leq \rho \end{cases} \tag{4.1}$$

if $p \neq N$ and

$$u_\rho(x) = \begin{cases} \dfrac{\log|x| - \log r}{\log\rho - \log r} & \text{if } \rho \leq |x| \leq r \\ 1 & \text{if } 0 \leq |x| \leq \rho \end{cases} \tag{4.2}$$

if $p = N$ (see [5,9]). Given $\rho \in (0, r)$ one can compute $\lambda = |\nabla u_\rho|$ for $|x| = \rho$, and then minimise λ. This yields the smallest possible value of λ such that (1.4) has a non-trivial solution. These optimal values have been calculated in [5] for $p = 2$ and in [9] for general $p \in (1, \infty)$. They are

$$\lambda_p(B_r) = \frac{\left|\frac{p-N}{p-1}\right|}{r\left|\left(\frac{p-1}{N-1}\right)^{(N-1)/(N-p)} - \left(\frac{p-1}{N-1}\right)^{(p-1)/(N-p)}\right|}$$

if $p \neq N$ and

$$\lambda_p(B_r) = \frac{e}{r}$$

if $p = N$. Unfortunately, the corresponding solution does *not* minimise $J_{\lambda_p,p}$. In case $p = 2$ this is pointed out in [5, Section 5.3], but also follows from the calculations below. To obtain $\Lambda_p(B_r)$ we start by computing $J_{\lambda,p}(u_\rho)$. We first consider the case $p \neq N$. An elementary calculation yields

$$|\nabla u_\rho(x)| = \Big|\frac{p-N}{p-1}\Big| \frac{|x|^{(1-N)/(p-1)}}{|\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}|}$$

for $\rho \leq |x| \leq r$ and zero elsewhere. Because

$$\begin{aligned}\int_\rho^r s^{p(1-N)/(p-1)} s^{N-1}\, ds &= \int_\rho^r s^{(p-N)/(p-1)-1}\, ds \\ &= \frac{p-1}{p-N}\big(r^{(p-N)/(p-1)} - \rho^{(p-N)/(p-1)}\big)\end{aligned}$$

we get

$$\int_{B_r} |\nabla u_\rho(x)|^p\, dx = \Big|\frac{p-N}{p-1}\Big|^{p-1} \frac{\omega_N}{|\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}|^{p-1}}, \tag{4.3}$$

where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. According to Theorem 1.1 we have to find the smallest possible $\lambda > 0$ such that

$$J_{\lambda,p}(u_\rho) = J_{\lambda,p}(0) = (p-1)\lambda^p |B_r| = (p-1)\frac{\omega_N}{N} r^N \lambda^p.$$

Using the definition of $J_{\lambda,p}$ and u_ρ we therefore require that

$$\begin{aligned}\Big|\frac{p-N}{p-1}\Big|^{p-1} &\frac{\omega_N}{|\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}|^{p-1}} \\ &+ (p-1)\lambda^p \frac{\omega_N}{N}(r^N - \rho^N) = (p-1)\frac{\omega_N}{N} r^N \lambda^p \end{aligned} \tag{4.4}$$

or equivalently

$$N\Big|\frac{p-N}{p-1}\Big|^{p-1} = (p-1)\lambda^p \rho^N \Big|\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}\Big|^{p-1}. \tag{4.5}$$

Clearly we get the smallest value of λ if we pick $\rho \in (0, r)$ such that

$$\rho^N \Big|\rho^{(p-N)/(p-1)} - r^{(p-N)/(p-1)}\Big|^{p-1}$$

is maximal, and then compute the corresponding value of λ from (4.5). An elementary calculation shows that this is the case for

$$\rho = \left(\frac{N}{p}\right)^{(p-1)/(p-N)} r, \tag{4.6}$$

and hence, if we substitute that value of ρ into (4.5), then

$$\Lambda_p(B_r) = \frac{p}{p-1}\left(\frac{p}{N}\right)^{(N-1)/(p-N)} \frac{1}{r}. \tag{4.7}$$

We could confirm the above by computing $|\nabla u_\rho|$ for the above value of ρ. If $p = N$ we proceed in exactly the same way to get

$$\rho = e^{-(1-1/N)} r \tag{4.8}$$

and

$$\Lambda_N(B_r) = \frac{N}{N-1} e^{(1-1/N)} \frac{1}{r}. \tag{4.9}$$

It is also evident that

$$\left(\frac{p}{N}\right)^{(N-1)/(p-N)} = \left(1 + \frac{\frac{1}{N}}{\frac{1}{p-N}}\right)^{(N-1)/(p-N)} \to e^{(N-1)/N}$$

as $p \to N$, so $\Lambda_p(B_r) \to \lambda_N(B_r)$ as $p \to N$. Also note that $\lambda_p(B_r) < \Lambda_p(B_r)$ for all $p \in (1, \infty)$. In particular, this proves the second part of Theorem 1.2.

Acknowledgement B.K. thanks Henrik Shahgholian for pointing out this problem during the open problem session of the ESF-sponsored conference "FBP 2008" in Stockholm. D.D. thanks for a very pleasant visit to the University of Cologne.

References

[1] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 105–144. MR618549 (83a:49011)

[2] A. Beurling, *On free-boundary value problems for the Laplace equation*, Seminars on analytic functions (Princeton, 1957), Vol. I, Institute for Advanced Study, Princeton, 1958, pp. 248–263.

[3] J. E. Brothers and W. P. Ziemer, *Minimal rearrangements of Sobolev functions*, J. Reine Angew. Math. **384** (1988), 153–179. MR929981 (89g:26013)

[4] D. Danielli and A. Petrosyan, *A minimum problem with free boundary for a degenerate quasilinear operator*, Calc. Var. Partial Differential Equations **23** (2005), no. 1, 97–124. MR2133664 (2006c:35303)

[5] M. Flucher and M. Rumpf, *Bernoulli's free-boundary problem, qualitative theory and numerical approximation*, J. Reine Angew. Math. **486** (1997), 165–204. MR1450755 (98i:35214)

[6] J. Frehse, *Capacity methods in the theory of partial differential equations*, Jahresber. Deutsch. Math.-Verein. **84** (1982), no. 1, 1–44. MR644068 (83j:35040)

[7] K. Friedrichs, *Über ein Minimumproblem für Potentialströmungen mit freiem Rande*, Math. Ann. **109** (1934), no. 1, 60–82. MR1512880

[8] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, Clarendon Press, New York, 1993. MR1207810 (94e:31003)

[9] A. Henrot and H. Shahgholian, *Existence of classical solutions to a free boundary problem for the p-Laplace operator. II. The interior convex case*, Indiana Univ. Math. J. **49** (2000), no. 1, 311–323. MR1777029 (2001m:35326)

[10] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Mathematics, vol. 1150, Springer-Verlag, Berlin, 1985. MR810619 (87a:35001)

[11] I. Ly and D. Seck, *Isoperimetric inequality for an interior free boundary problem with p-Laplacian operator*, Electron. J. Differential Equations **2004** (2004), no. 109, 1–12. MR2108880 (2005h:35374)

[12] S. Martínez and N. Wolanski, *A minimum problem with free boundary in Orlicz spaces*, Adv. Math. **218** (2008), no. 6, 1914–1971. MR2431665 (2009h:35456)

[13] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372. MR0463908 (57 #3846)

Author's addresses:

Daniel Daners
School of of Mathematics and Statistics
The University of Sydney, NSW 2006, Australia
daniel.daners@sydney.edu.au

and

Bernd Kawohl
Mathematisches Institut
Universität zu Köln
D-50923 Köln, Germany
kawohl@mi.uni-koeln.de