WiSe exercises

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Exercise 1

Let $n \in \mathbb{Z}_{\geq 0}$ have digits $c_0, \ldots, c_d$ in base p. Check directly that

$$(c_0 + 1)(c_1 + 1) \cdots (c_d + 1) \leq n + 1,$$

and determine when equality holds. Explain this inequality as a shadow of a theorem in the modular representation theory of SL_2, and use it to state precisely for which values of n the module ∇_n is simple.

Exercise 2

Let k be a field and $V \neq 0$ a finite-dimensional k-vector space. Work out why the natural map

$$\mathrm{Sym}^n(V^*) \to (\mathrm{Sym}^n(V))^*$$

is an isomorphism if and only if $n! \neq 0$ in k.

Exercise 3

Throughout this exercise, k is an algebraically closed field; in this exercise and later ones, all representations are assumed to be finite dimensional.

(a) Let G be a reduced algebraic k-group. Discuss the natural Hopf algebra structure on the regular functions $k[G]$ and the equivalence of abelian tensor categories,

$$\{\text{representations of } G\} \cong \{\text{left } G\text{-modules}\} \cong \{\text{right } k[G]\text{-comodules}\}.$$

(b) Using these equivalences, show that for a torus T we can identify

$$\mathrm{Rep}(T) \cong \{X(T)\text{-graded } k\text{-vector spaces}\}.$$

(c) Show that in characteristic zero, we can identify

$$\mathrm{Rep}(\mathbb{G}_a) \cong \{(V, \phi) : V \text{ a } k\text{-vector space, } \phi \in \mathrm{End}_k(V) \text{ nilpotent}\},$$

while in characteristic p, the right-hand side must instead be

$$\{(V, \phi_n)_{n \geq 1} : V \text{ a } k\text{-vector space, } \phi_i \in \mathrm{End}_k(V) \text{ all commuting, } \phi_i^p = 0\}.$$

Exercise 4

Let k be a field and G a reductive algebraic group over k with maximal torus T, Weyl group $W = N_G(T)/T$, and $X = X(T)$.

(a) Check that the character of a G-module is invariant under the W-action on $\mathbb{Z}[X]$ induced by the W-action on X, i.e. $w \cdot e^{\lambda} = e^{w(\lambda)}$ for $w \in W$.
(b) Hence argue that if M is a G-module with $\dim M_\lambda = 1$ and all weights W-conjugate to $\lambda \in X_+$, then $M \cong L(\lambda)$ is simple.
(c) Let $G = \mathrm{GL}_n$ have natural representation V of dimension n. Prove by considering weights that $\bigwedge^m V$ is simple for $1 \le m \le n$.

Exercise 5

(a) Let $v_p(n)$ be the p-adic valuation of $n \in \mathbb{Z}_{\ge 1}$. Prove *Legendre's formula*:

$$v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

(b) By writing n in base p, rephrase Legendre's formula as

$$v_p(n!) = \frac{n - s_p(n)}{p-1}, \quad \text{where} \quad s_p(n) = \text{sum of } p\text{-adic digits of } n.$$

(c) For $m \in \mathbb{Z}_{\ge 1}$, write

$$\begin{aligned} m + n &= a_d p^d + a_{d-1} p^{d-1} + \cdots + a_0, \\ m &= b_d p^d + b_{d-1} p^{d-1} + \cdots + b_0, \\ n &= c_d p^d + c_{d-1} p^{d-1} + \cdots + c_0; \end{aligned}$$

there may be some leading zeroes. For $0 \le i \le d$, the indicator $\gamma_i \in \{0, 1\}$ of the i-th carry when adding m to n is defined inductively as follows:

$$\gamma_0 = \begin{cases} 0 & \text{if } b_0 + c_0 < p \\ 1 & \text{otherwise,} \end{cases} \qquad \gamma_i = \begin{cases} 0 & \text{if } b_i + c_i + \gamma_{i-1} < p \\ 1 & \text{otherwise.} \end{cases}$$

Using (b) and the expression $\binom{m+n}{n} = \frac{(m+n)!}{m!n!}$, prove *Kummer's theorem*:

$$v_p \binom{m+n}{n} = \sum_{i=0}^{d-1} \gamma_i.$$

(d) Deduce that if $n+1$ has only one non-zero digit in base p, then p does not divide $\binom{n}{j}$ for all $0 \le j \le n$. What does this have to do with Exercise 1?

References and accreditation

- Exercise 2 is inspired by these notes of Brian Conrad.
- I learned Exercise 3 from Geordie Williamson.
- Exercise 4 is constructed from Section II.2.15 of Jens Carsten Jantzen's *Representations of Algebraic Groups*.
- The approach for Parts (a)–(c) of Exercise 5 is due to Karen Ge.