Rigorously validated estimation of statistical properties of expanding maps

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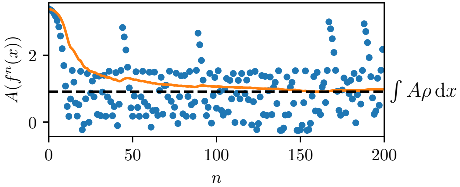
Introduction

Chaotic systems are commonly studied in terms of their ergodic theory:



$$\frac{1}{N}\sum_{i=0}^{N-1} A(f^n(x)) \xrightarrow{N\to\infty,\, x \text{ a.e.}} \int A\rho\,\mathrm{d}x$$

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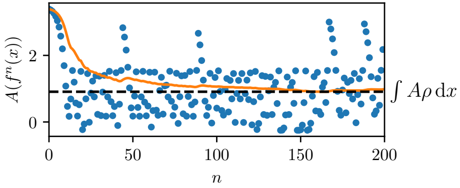

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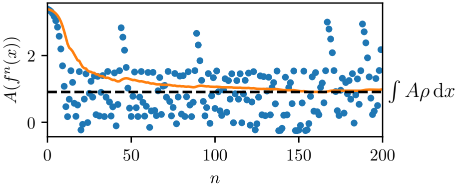

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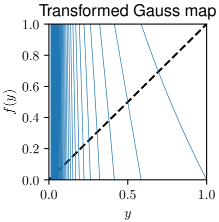
Introduction

We consider subclass of chaotic maps: full-branch uniformly expanding maps.

- Simple, illustrative model class
- Contains examples
  of independent theoretical interest,
  e.g. Gauss map on $[0,1]$ (continued fractions), $f(x) = x^{-1} \mod 1$
  under change of variable $2^y - 1 = x$



Rigorous numerics can answer various theoretical problems (e.g. dimensionality of Lagrange and Markov spectra).

Introduction

Current methods:

- Dynamical zeta methods (Pollicott, Jenkinson, *et al.*): only practical with a few branches, assumes maps are analytic
- Ulam's method on transfer operators (Galatolo, Nisoli, *et al.*): low-regularity method, only obtain a couple of rigorous digits.

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Goal: powerful numerics for a broad range of maps.

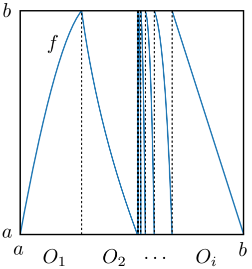
We will use a *Chebyshev Galerkin method* for *transfer operators*.

Maps under consideration

We consider maps of the interval $f : [a, b] \circlearrowleft$ with nice properties:

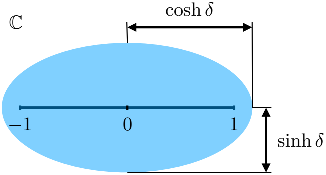
- Countable partition $\overline{\cup_{i \in I} O_i} = [a, b]$, $f|_{O_i}$ bijections with inverses v_i
- Regularity conditions on distortion $D_i := \log |v_i'|$, either:
  - $\sup_{s \leq r, i \in I} \|D_i^{(r)}\|_\infty \leq B_{D,r}$ for some r;
  - $\sup_{z \in \mathrm{Bern}(e^\delta), i \in I} |D_i(z)| \leq B_{A,\delta}$, where $\mathrm{Bern}(e^\delta)$ is a Bernstein ellipse...
- Technical requirement on placement of the O_i.
- Uniform "C-expansion" condition:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos^{-1} \circ f \circ \cos)(\theta) \geq \check{\gamma} > 1.$$



Spectral basis 2: Chebyshev series

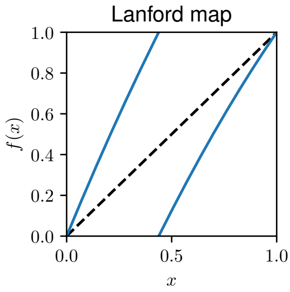
A Bernstein ellipse of parameter e^{δ} is cos :



Lanford map

Standard map in this class used to test numerics is the Lanford map (e.g. Jenkinson *et al.*, '18, Bahsoun *et al.*, '16):

$$f(x) = 2x + \frac{1}{2}x(1-x) \mod 1$$



Transfer operator

We use the so-called transfer operator $\mathcal{L} : \mathcal{B} \circlearrowleft$.
This tracks the action of the map f on signed measure densities in some Banach space $\mathcal{B}$ of smooth functions:

$$\int_a^b A \circ f \, \varphi \, dx = \int_a^b A \, \mathcal{L}\varphi \, \mathrm{d}x.$$

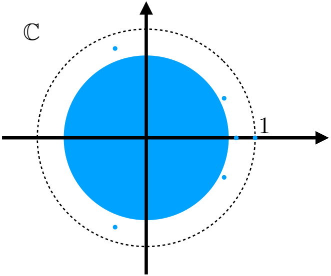
Explicit formula for pointwise evaluation:

$$(\mathcal{L}\varphi)(x) = \sum_{i \in I} \sigma_i v_i'(x) \varphi(v_i(x)),$$

where v_i are the inverses of $f|_{O_i}$, and $\sigma_i = \operatorname{sign} v_i'$.
(Weights other than $\sigma_i v_i'$ also useful in various situations.)

Transfer operator: functional analysis

The transfer operator is *quasi-compact* on a range of Banach spaces, and in particular always has an isolated eigenvalue at 1:



We will use as our Banach space $\mathcal{B} = BV$, the space of functions of bounded variation.

NB: $\|\phi\|_{BV} = TV(\phi) + \|\phi\|_\infty$ and $\|\phi\psi\|_{BV} \leq \|\phi\|_{BV}\|\psi\|_{BV}$.

Transfer operator

Our particular quantities of interest can be expressed using resolvent data at eigenvalue 1

- Absolutely continuous invariant measure $\rho \mathrm{d}x$ satisfies

$$\left\{ \begin{array}{rcl} \mathcal{L}\rho & = & \rho, \\ \mathscr{S}\rho & = & 1, \end{array} \right.$$

  where $\mathscr{S}\varphi := \int_b^a \varphi \, \mathrm{d}x$.

- Diffusion coefficient $\sigma_f^2(A)$ satisfies

$$\sigma_f^2(A) = \mathscr{S}\left[A \sum_{i=-\infty}^{\infty} \mathcal{L}^{|i|}(\rho A - \rho \mathscr{S}[\rho A]) \right]$$

In general, no explicit solutions!

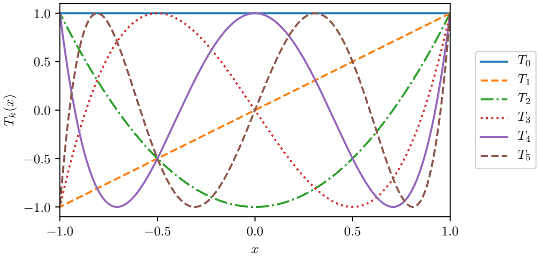
Chebyshev series

We will use as our approximation basis the Chebyshev polynomials on $[-1, 1]$:

$$T_k(x) = \cos(k \cos^{-1} x), k = 0, 1, \ldots$$

These are related to Fourier series via transformation $x = \cos\theta$. Orthogonality relation

$$\int_{-1}^{1} T_k(x) T_j(x) \frac{\mathrm{d}x}{\sqrt{1 - x^2}} = t_k^{-1} \pi \delta_{jk}$$



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$t_k = 2 - \delta_{0k}$

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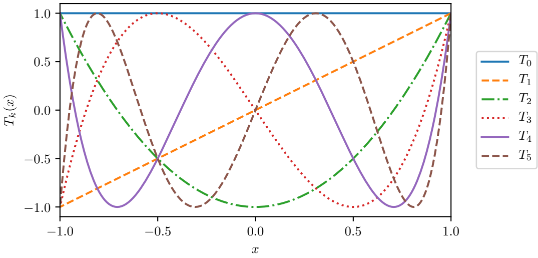
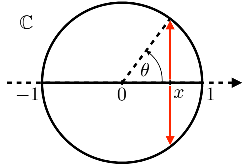


Chebyshev series

Take a C^1 function $\psi : [-1,1] \to \mathbb{R}$.
Via orthogonality have



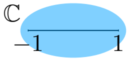
$$\psi(x) = \sum_{k=0}^{\infty} \check{\psi}_k T_k(x),$$

where

$$\check{\psi}_k = \frac{t_k}{\pi} \int_{-1}^{1} \psi(x) T_k(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}}.$$

Using Fourier series connection, we find $|\check{\psi}_k| = \mathcal{O}(s(k))$, where the spectral convergence rate

$$s(k) := \begin{cases} k^{-r}, & \psi \in C^r \\ e^{-\delta k}, & \psi \text{ bd. and analytic on} \\ & e^{\delta}\text{-Bernstein ellipse} \end{cases}$$



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Transfer operator in Chebyshev basis

What do we get if we write the transfer operator as acting on Chebyshev coefficients?
That is, consider infinite matrix of $\mathcal{L}_{jk}, j, k = 0, 1, 2, \ldots$:

$$\mathcal{L}_{jk} = \frac{t_j}{\pi} \int_{-1}^{1} (\mathcal{L} T_k)(x)\, T_j(x) \, \frac{\mathrm{d}x}{\sqrt{1 - x^2}}$$

so

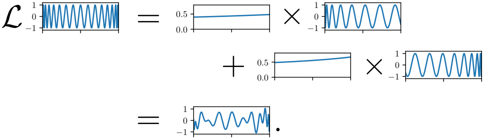
$$\mathcal{L} T_k = \sum_{j=0}^{\infty} \mathcal{L}_{jk} T_j .$$

Transfer operator in Chebyshev basis

The transfer operator sends oscillating functions to functions of lower frequency:

$$\mathcal{L} T_k = \sum_{i \in I} (\sigma_i v_i') \times (T_k \circ v_i).$$

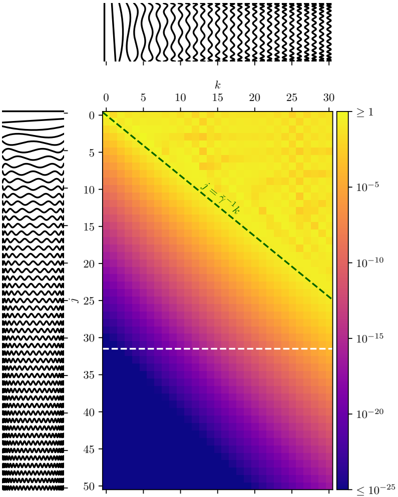
Graphically,



Thus expect that $\mathcal{L}_{jk} \ll 1$ for $j > k$.
Can prove this using oscillatory integral techniques on orthogonality relation.

Transfer operator in Chebyshev basis

"Heat map" of $|\mathcal{L}_{jk}|$:



Transfer operator in Chebyshev basis

Theorem (W. '19)

For all $p > \check{\gamma}^{-1}$ there exists C depending on regularity of distortion D_i such that

$$|\mathcal{L}_{jk}| \leq C \min\{1, s(j - pk)\},$$

where s is the spectral convergence rate of the map f.

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where s *is the spectral convergence rate of the map* f.

Upshot: the transfer operator is close to "upper-triangular + finite-rank"

Galerkin method

Take a family of finite-rank projections $\mathcal{P}_N : BV \circlearrowleft$ which asymptotically approximate the identity.
Pick large(ish) N:

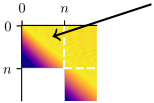
- Compute the finite-dimensional operator $\mathcal{L}_N := \mathcal{P}_N \mathcal{L}_N|_{\operatorname{im} \mathcal{P}_N}$.
- Substitute $\mathcal{P}_N \mathcal{L}|_{\operatorname{im} \mathcal{P}_N}$ for $\mathcal{L}$ in the problem of interest, e.g. for acim $\mathcal{P}_N \mathcal{L} \rho_N = \rho_N$.
- Numerically solve to get estimate: e.g. ρ_N should approximate true acim ρ.

In our case, $\mathcal{P}_N$ is projection onto Chebyshev modes $T_0, \dots T_N$.

Operator approximation

Our finite rank operator $\mathcal{L}_N$ is the top-left block of a block-upper triangular operator

$$\tilde{\mathcal{L}}_N := \mathcal{L} - (\mathrm{id} - \mathcal{P}_N)\mathcal{L}\mathcal{P}_N =$$



$\mathcal{L}_N := \mathcal{P}_N \mathcal{L}|_{\mathrm{im}\,\mathcal{P}}$

In particular, $\tilde{\mathcal{L}}_N|_{\mathrm{im}\,\mathcal{P}_N} = \mathcal{L}_N$.

Theorem (W. '19)

There exists a constant C depending on bounds on the D_i such that

$$\|\tilde{\mathcal{L}}_N - \mathcal{L}\|_{BV} \leq CN^{1+\epsilon} s(N)\},$$

where s is the spectral convergence rate of the map f.

Solution operator

We want to probe the resolvent data of $\mathcal{L}$ at eigenvalue 1 (for acim, diffusion coefficient, etc.). To do this we will use the *solution operator*:

$$\mathcal{S} = (\mathrm{id} - \mathcal{L} + 1\mathscr{S})^{-1}$$

resolvent of $\mathcal{L}$ — rank 1 perturbation with left eig'f'n $\mathscr{S}$

Has useful properties:

- $\mathcal{S}1 = \rho$
- $\mathcal{S}\varphi = \sum_{k=0}^{\infty} \mathcal{L}^k \varphi$ if $\int_{-1}^{1} \varphi \, \mathrm{d}x = 0$
  - Can write $\sigma_f^2(A) = \mathscr{S}\,[A\,(2\mathcal{S} - \mathrm{id})(\rho\, A - \rho\, \mathscr{S}[\rho\, A])]$

Convergence of estimates: operator error

Then, since $\mathcal{S}$ is just an operator function of $1\mathscr{S}$ (which is upper-triangular and whose Chebyshev coefficients we know) and $\mathcal{L}$, if

$$\tilde{\mathcal{S}}_N := (\operatorname{id} - \tilde{\mathcal{L}}_N + 1\mathscr{S})^{-1}$$

then

$$\|\tilde{\mathcal{S}}_N - \mathcal{S}\|_{BV} = \mathcal{O}(N^{1+\epsilon}\, s(N)),$$

and by block-upper-triangularity we can compute $\tilde{\mathcal{S}}_N|_{\operatorname{im}\mathcal{P}_N} = (\operatorname{id} - \mathcal{L}_N + 1\mathscr{S}|_{\operatorname{im}\mathcal{P}_N})^{-1}$.

Convergence of estimates: operator error

Theorem (W. '19)

There exist constants C, C' depending on bounds on the D_i such that if $N^{1+\epsilon} s(N) \|\mathcal{S}\|_{BV} \leq C'$ then

$$\|\tilde{\mathcal{S}}_N - \mathcal{S}\|_{BV} \leq C \|\mathcal{S}\|_{BV} N^{1+\epsilon} s(N)\},$$

where s is the spectral convergence rate of the map f.

Once again, our Galerkin approximation $\mathcal{S}_N = \tilde{\mathcal{S}}_N|_{\mathrm{im}\,\mathcal{P}_N}$.

(NB: also possible to use bounds on $|\mathcal{L}_{jk}|$ for estimates in the style of Keller and Liverani '99)

Bounds on norm of $\mathcal{S}$

How to estimate norm of solution operator $\|\mathcal{S}\|_{BV}$?

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- But, if our map is analytic we have exponential convergence of the estimates and this is fine.
- A posteriori bounds are also possible (Galatolo and Nisoli '14).

How to compute $\mathcal{L}_N$?

Our transfer operator approximation $\mathcal{L}_N$, acting on Chebyshev coefficients, is just the first $(N+1) \times (N+1)$ block of $\mathcal{L}$. But recall the coefficients of $\mathcal{L}_{jk}$ are

$$\mathcal{L}_{jk} = \frac{t_j}{\pi} \int_{-1}^{1} (\mathcal{L}T_k)(x)\, T_j(x)\, \frac{\mathrm{d}x}{\sqrt{1-x^2}},$$

i.e. we need to compute the first $N+1$ Chebyshev coefficients of each $\mathcal{L}T_k, k = 0, 1, \ldots N$.

We can estimate coefficients $\mathcal{L}_{jk}$ very accurately via interpolating $\mathcal{L}T_k$ on $M > N$ *Chebyshev points*

$$x_{l,M} = \cos \frac{2l+1}{M}\pi,\ l = 1, \ldots, M.$$

Fast algorithm for this based on Fast Fourier Transform.

How to compute $\mathcal{L}_N$?

Error given by aliasing formula:

$$\mathcal{L}_{jk} = \mathcal{L}_{jk}^M - \underbrace{\sum_{l=1}^{\infty} \mathcal{L}_{2lM-j,k} + \mathcal{L}_{2lM+j,k}}_{\text{rigorously bounded a priori}}$$

Thus get rigorous interval estimate for the $\mathcal{L}_{jk}$ that is very efficient to compute.

Rigorous algorithm

To compute invariant measures:

- Generate $\mathcal{L}_N = (\mathcal{L}_{jk})_{j,k=0,\ldots,N}$ using pointwise evaluation of transfer operator and FFT algorithm ($O(N^2 \log N)$)
- Estimate $\mathcal{S}_N = (I - \mathcal{L}_N + (1\mathscr{S})_N)^{-1}$ ($O(N^3)$ but reusable)
- Compute coefficients of $\rho_N := \mathcal{S}_N 1$.
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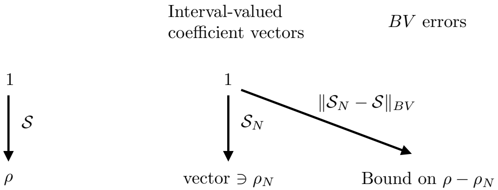
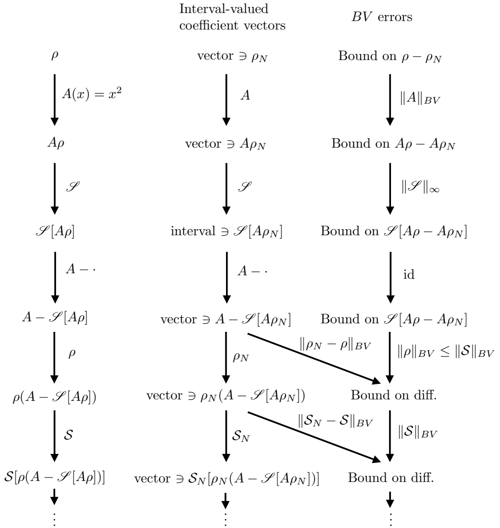
Output: a vector of intervals containing the coefficients of ρ_N, plus a bound on the BV norm of $\rho - \rho_N$.
Can of course roll interval diameters into BV error.

Rigorous algorithm



Rigorous algorithm

$$\sigma_f^2(A) = \mathscr{S}\left[A\,(2\mathcal{S} - \mathrm{id})(\rho\,A - \rho\,\mathscr{S}[\rho\,A])\right]$$



Validated bounds

Using $N = 2048$ we have:

Theorem (W. '19)

a. *The Lanford map's Lyapunov exponent* $L_{exp} := \int_\Lambda \log|f'|\,\rho\,dx$ *lies in the range*

$$L_{exp} = 0.657\,661\,780\,006\,597\,677\,541\,582\,413\,823\,832\,065\,743\,241\,069\,580\,012\,201\,953\,952\,802\,691\,632\,666\,111\,554\,023\,759\,556\,459\,752\,915\,174\,829\,642\,156\,331\,798\,026\,301\,488\,594\,89 \pm 2 \times 10^{-128}.$$

b. *The diffusion coefficient for the Lanford map with observable* $A(x) = x^2$ *lies in the range*

$$\sigma_f^2(A) = 0.360\,109\,486\,199\,160\,672\,898\,824\,186\,828\,576\,749\,241\,669\,997\,797\,228\,864\,358\,977\,865\,838\,174\,403\,103\,617\,477\,981\,402\,783\,211\,083\,646\,769\,039\,410\,848\,031\,999\,960\,664\,7 \pm 6 \times 10^{-124}.$$

Related results

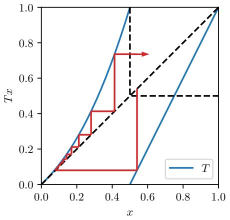
- Slipantschuk and Bandtlow ('20): using Chebyshev approximation of analytic expanding maps, all eigendata converge exponentially.
- Bandtlow *et al.* ('20): Chebyshev approximation of expanding maps used to compute Laplace operator spectra of some infinite hyperbolic surfaces
- Crimmins and Froyland ('19): statistical properties that are functions of the transfer operator (e.g. large deviations) can be estimated using transfer operator discretisations.

Intermittent systems

Interested in statistical properties of *non*-uniformly expanding maps, for example $T : [0,1] \circlearrowleft$

$$Tx = \begin{cases} x(1 + 2^\alpha x^\alpha), & x \leq \frac{1}{2}, \\ 2x - 1, & x > \frac{1}{2}, \end{cases}$$

where $\alpha > 0$.



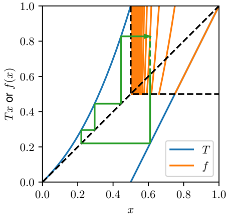
- Lack of uniform expansion and weak mixing properties makes numerics very challenging.
- Ulam-style methods very slow, non-viable for $\alpha \gg 1$ (infinite ergodic theory).

Intermittent systems

We approach via induced map $f : [\frac{1}{2}, 1] \circlearrowleft$:

$$f(x) := T^{\tau_T(x)} x.$$

This map is analytic and full-branch uniformly-expanding: we can use Chebyshev methods on it.



However, induced map f

- (a) is difficult to compute when $\tau_T \gg 1$, and
- (b) has an infinite number of branches (problem for computing transfer operator)

To solve (a):

Theorem (W., forthcoming)

There exists a real-analytic function $A : (0,1] \to [0,\infty)$ *such that*

$$f(x) = A^{-1}(A(T(x)) \mod 1),$$

The function A has an asymptotic expansion near 0 with explicit bounds on error.

To solve (b), note that transfer operator of f is

$$(\mathcal{L}_f \varphi)(x) = \sum_{i=0}^{\infty} \frac{A'(x)}{2} \frac{\varphi(A^{-1}(A(2x-1)+i))}{A'(A^{-1}(A(2x-1)+i))}.$$

We can use smoothness to solve this! When $\varphi = T_k$ (i.e. smooth), can use Euler-Maclaurin formula. (One can also do this with the Gauss map.)

Intermittent systems

Effective estimates of statistical properties of both the induced map and the full, non-uniformly expanding map, for a wide range of α:

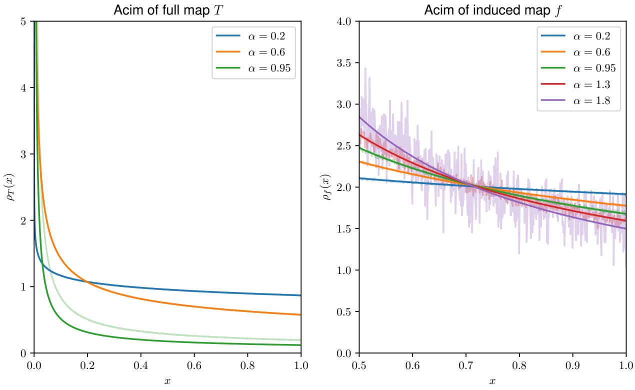


Figure: Acims of the full map with different normalisations. Pale colours indicate estimates from binning on 10^8 simulations.

Upshot: very accurate validated bounds again possible.
For example, using $N = 512$, the expected return time to $[1/2, 1]$ for $\alpha = 0.95$ (a near-singular case) lies in the range

$$\mathbb{E}_f[\tau_T] = 14.073\,323\,220\,001\,939\,529\,241\,549\,699\\ 610\,756\,609\,803\,3171 \pm 10^{-43}.$$

Conclusion

Chebyshev Galerkin transfer operator discretisations provide a very effective way to rigorously estimate statistical properties of full-branch uniformly expanding maps.

Some further directions:

- Better/higher-order function spaces (e.g. for estimating first-order response to perturbations)
- A posteriori estimates on decay of correlations (i.e. on $\|\mathcal{S}\|$)
- General spectral data

Wormell, C.L., Spectral Galerkin methods for transfer operators in uniformly expanding dynamics. *Numerische Mathematik* 142 (2019) 421–463.