Symmetric groups and related algebras

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There exist many unexpected relations between the representation theory of the symmetric group and the representation theory of other algebras.

One of the most tantalizing problems concerning the symmetric groups is to determine the dimensions of their irreducible modules. Over fields of characteristic zero, the solution is well known, since for each partition λ, there exists a Specht module S^λ, and the Specht modules provide a complete set of irreducible modules. However, a Specht module S^λ can be defined over an arbitrary field, and its dimension is equal to the number of standard λ-tableaux. If one could properly understand the structure of S^λ then the dimensions of the irreducible modules for the symmetric group would be known.

A good way to define Specht modules involves an endomorphism algebra which is called the Schur algebra. It turns out that the problem of determining the dimensions of the irreducible modules for all Schur algebras is equivalent to two other important problems, namely

(i) determining the dimensions of the irreducible modules for all symmetric groups; and

(ii) determining the dimensions of the irreducible modules for all general linear groups in the describing characteristic.

Remarkably, these problems, in turn, are special cases of a third problem:

(iii) determining the dimensions of the irreducible modules for all general linear groups $\mathrm{GL}_n(q)$ in a non-describing characteristic.

The connection is provided by the q-Schur algebra, which is the same as the usual Schur algebra if $q = 1$.

Recently, a significant advance towards a solution of all these problems has been provided by the proof (by Ariki – see also Grojnowski) of the Lascoux-Leclerc-Thibon Conjecture.