A comment on separation properties

Earlier in the course the following definitions were made:

(a) A topological space X is said to be T_1 if the following condition holds: for all a, $b \in X$, if $a \neq b$ then there exists an open set U with $a \in U$ and $b \notin U$.

(b) A topological space X is said to be T_2, or Hausdorff, if the following condition holds: for all a, $b \in X$, if $a \neq b$ then there exists an open set U with $a \in U$ and an open set V with $b \in V$, such that $U \cap V = \emptyset$.

There is perhaps a possible source of confusion relating to T_1, since the definition given in Choo's notes may at first sight seem to be different from the definition above, though in fact the two are equivalent. Choo also gives the following definition: a space is said to be T_0 if for every pair of distinct points in the space there is an open set containing one and not the other. To make it clear that this concept is different from T_1, it is helpful to rephrase the definition: a space X is T_0 if for all x, $y \in X$ with $x \neq y$, there either exists an open set U containing x and not y or an open set V containing y and not x. If the space is T_1 and x, y are distinct points of X then, applying the definition above with $a = x$ and $b = y$, we see that there must exist an open set U containing x and not y; moreover, applying the same condition with $a = y$ and $b = x$ we see that there must also exist an open set V containing b and not a. Choo chose to emphasize this by stating the definition of T_1 as follows: X is called a T_1-space if for every pair a, b of distinct points of X there are open sets U and V in X such that $a \in U$ but $b \notin U$ and $b \in V$ but $a \notin V$. The comments above show that this stronger seeming requirement is actually no stronger than the definition I gave. Mine implies his, and his obviously implies mine.

Choo chose to reword the definition of T_1 to emphasize its difference from T_0. However, the definition as I have given it above is simpler and clearer; it was the definition of T_0 that really needed clarification.

Before leaving this, it is as well to recall that T_1 could be defined as follows: X is T_1 if, for all $x \in X$, the set $\{x\}$ is closed. Proving the equivalence of this definition with the one above is an easy exercise, which the reader should now do mentally. (The proof can be found in an earlier lecture if need be.)

For interest only

To end these lectures I would like to mention an interesting construction known as the "one point compactification" of a topological space X. The idea is this. Given a topological space X we wish to construct another space X', containing X as a subspace, and such that X' is compact. Furthermore, X' should contain just one point that is not in X.

Let $\mathcal{T} = \{\mathcal{U} \subseteq \mathcal{X} \mid \mathcal{U}$ is open $\}$, the given topology on the set X. We need to add an extra point to X; let us call this new point ∞. Now define $\mathcal{T}'$ to be the set of all subsets V of X' such that either

(1) $V \subseteq X$ and $V \in \mathcal{T}$, or

(2) $V = \{\infty\} \cup (X \setminus C)$ for some $C \subseteq X$ such that C is compact and closed relative to the topology $\mathcal{T}$.

The proof that $\mathcal{T}'$ is a topology on X' makes use of some of the elementary properties of compact sets that were discussed in Tutorial 10. It is slightly easier in this case to work with closed sets rather than open sets; so let us define $\mathcal{S} = \{ F \subseteq X \mid X \setminus F \in \mathcal{T} \}$, the closed sets of the given topology on X, and let $\mathcal{S}' = \mathcal{S}'_1 \cup \mathcal{S}'_2$, where

(1′) $\mathcal{S}'_1 = \{ \{\infty\} \cup F \mid F \in \mathcal{S} \}$, and

(2′) $\mathcal{S}'_2 = \{ C \subseteq X \mid C \text{ is compact and closed relative to the topology } \mathcal{T} \}$.

We need to check that $\emptyset$, $X' \in \mathcal{S}'$, the union of two elements of $\mathcal{S}'$ is in $\mathcal{S}'$, and the intersection of an arbitrary collection of elements of $\mathcal{S}'$ is in $\mathcal{S}'$. It is immediately apparent that $X' \in \mathcal{S}'_1$, since $X \in \mathcal{S}$, and $\emptyset \in \mathcal{S}'_2$ since it is straightforward to show that the empty set is compact (and closed) in any topology.

Since the union of two elements of $\mathcal{S}$ is always an element of $\mathcal{S}$, the union of two elements of $\mathcal{S}'_1$ is in $\mathcal{S}'_1$, and the union of an element of $\mathcal{S}'_1$ with one of $\mathcal{S}'_2$ gives an element of $\mathcal{S}'_1$. Furthermore, as shown in Exercise 2 of Tutorial 10, the union of two compact sets is always compact; so the union of two elements of $\mathcal{S}'_2$ gives an element of $\mathcal{S}'_2$. Hence $\mathcal{S}'$ is closed under unions of pairs of elements (and hence under finite unions).

Now consider the intersection of an arbitrary family $(G_i)_{i \in I}$ of elements of $\mathcal{S}'$. If every G_i in this family is in $\mathcal{S}'_1$, then we can write $G_i = \{\infty\} \cup F_i$, where $(F_i)_{i \in I}$ is a family of elements of $\mathcal{S}$. In this case we find that $\bigcap_{i \in I} G_i = \{\infty\} \cup \bigcap_{i \in I} F_i$, which is in $\mathcal{S}'_1$ since $\mathcal{S}$ is closed under arbitrary intersections. If one or more of the G_i's is in $\mathcal{S}'_2$ then $\infty \notin \bigcap_{i \in I} G_i$, and so $\bigcap_{i \in I} G_i = \bigcap_{i \in I} F_i$, where $F_i = X \cap G_i$ is a closed subset of X for each i, and at least one F_i is compact. Since $\mathcal{S}$ is closed under arbitrary intersections, we know that $\bigcap_{i \in I} F_i$ is closed. If we choose i_0 such that F_{i_0} is compact, then we see that $\bigcap_{i \in I} F_i = F_{i_0} \cap \bigcap_{i \in I} F_i$ is the intersection of a compact set and a closed set, hence is compact by Exercise 4 of Tutorial 10. So in this case $\bigcap_{i \in I} G_i$ is in $\mathcal{S}'_2$, and hence $\mathcal{S}'$ is closed under arbitrary intersections.

We have now shown that $\mathcal{T}'$ is a topology on X'. It is clear that the intersection of an element of $\mathcal{T}'$ with X always gives an element of $\mathcal{T}$, and, furthermore, every element of $\mathcal{T}$ arises in this way since $\mathcal{T} \subseteq \mathcal{T}'$. So the topology on X induced by the topology $\mathcal{T}'$ on X' coincides with the original topology $\mathcal{T}$. So the one remaining thing to be proved is that X' is compact with respect to the topology $\mathcal{T}'$.

Let $(V_i)_{i \in I}$ be a family of open subsets of X' whose union is X'. At least one of the sets V_i must contain the point ∞; so there exists $j \in I$ such that

$$V_j = \{\infty\} \cup (X \setminus C),$$

where C is a compact subset of X. Now we have that

$$C = X \cap C \subseteq X \cap \bigcup_{i \in I} V_i = \bigcup_{i \in I} (X \cap V_i),$$

and since $X \cap V_i \in \mathcal{T}$ for each i we see that $(X \cap V_i)_{i \in I}$ is an open covering of the compact subset C of X. Hence there exists a finite subset $I_0 \subseteq I$ such that $C \subseteq \bigcup_{i \in I_0} (X \cap V_i)$. Now $J = j \cup I_0$ is a finite subset of I, and we find that $(V_i)_{i \in J}$ is a finite subcovering of the original open covering of X', since

$$X' = V_j \cup C \subseteq V_j \cup \bigcup_{i \in I_0} (X \cap V_i) \subseteq V_j \cup \bigcup_{i \in I_0} V_i = \bigcup_{i \in J} V_i.$$

So we have shown, as required, that an arbitrary open covering of X' always has a finite subcovering. That is, X' is compact.

Note that a special case of this construction appeared in one of the questions of the 1999 exam. There the space X was taken to be the positive integers, with the discrete topology, and the point in X' and not in X was called 0 rather than ∞. Recall that in the discrete topology all subsets are open. It is easy to show that under these circumstances a subset is compact if and only if it is finite. So the upshot of this is that the topology on X' consisted of all subsets that do not contain 0 together with the cofinite sets that do contain 0. The exam question asked candidates to prove that this is indeed a topology, and that the space is compact. Of course, the necessary proofs are essentially the same as those given above, although they simplify somewhat in this special case.

THE END