Topological spaces were invented because they provide a natural context in which continuity can be defined and discussed. As we have seen, the concept of an "open set" is the key ingredient in this. Since compactness is defined in terms of open sets, it is natural to investigate relationships between compact sets and continuous functions.

Proposition. *Let X and Y be topological spaces and $f\colon X \to Y$ a continuous mapping. If C is any compact subset of X then $f(C)$ is a compact subset of Y.*

Proof. Let $(V_i)_{i\in I}$ be any open covering of $f(C)$. Then $f^{-1}(V_i) = \{\, x \in X \mid f(x) \in V_i \,\}$ is open in X, for each $i \in I$, since V_i is open in Y and f is continuous. If $c \in C$ is arbitrary, then $f(c) \in f(C) \subseteq \bigcup_{i\in I} V_i$, and so $f(c) \in V_i$ for some $i \in I$. Hence $c \in f^{-1}(V_i)$ for some $i \in I$, and thus $c \in \bigcup_{i\in I} f^{-1}(V_i)$. As $c \in C$ was arbitrary, this shows that $C \subseteq \bigcup_{i\in I} f^{-1}(V_i)$. So $\big(f^{-1}(V_i)\big)_{i\in I}$ is an open covering of C, and since C is compact there is a finite subset J of I such that $C \subseteq \bigcup_{i\in J} f^{-1}(V_i)$.

Now let $y \in f(C)$ be arbitrary. Then $y = f(x)$ for some $x \in C$, and since $C \subseteq \bigcup_{i\in J} f^{-1}(V_i)$ we have $x \in f^{-1}(V_i)$ for some $i \in J$. That is, $f(x) \in V_i$ for some $i \in J$. Thus we have shown that $y = f(x) \in \bigcup_{i\in J} V_i$, and since y was an arbitrary element of $f(C)$ we conclude that $f(C) \subseteq \bigcup_{i\in J} V_i$. Thus, starting from an arbitrary open covering $(V_i)_{i\in I}$ of $f(C)$ we have produced a finite subcovering. So $f(C)$ is compact, as claimed. □

Proposition. *Let X be a topological space and Y a subspace of X. If $C \subseteq Y$ then C is compact as a subset of X if and only if it is compact as a subset of Y.*

Proof. Suppose that C is compact as a subset of X; we shall show that it is compact as a subset of Y. Let $(V_i)_{i\in I}$ be an arbitrary family of subsets of Y such that $C \subseteq \bigcup_{i\in I} V_i$ and each V_i is an open subset of Y. By the definition of the subspace topology, for each $i \in I$ there is an open subset W_i of X such that $V_i = Y \cap W_i$. Then

$$C \subseteq \bigcup_{i\in I} V_i \subseteq \bigcup_{i\in I} W_i$$

and since C is compact in X it follows that there is a finite subset J of I such that $C \subseteq \bigcup_{i\in J} W_i$. Now since $C \subseteq Y$ we find that

$$C = Y \cap C \subseteq Y \cap \bigcup_{i\in J} W_i = \bigcup_{i\in J} (Y \cap W_i) = \bigcup_{i\in J} V_i.$$

So the open covering $(V_i)_{i\in I}$ of C has a finite subcovering, namely $(V_i)_{i\in J}$. Since this holds for all coverings of C by open subsets of Y, we have shown that C is compact as a subset of Y.

Conversely, suppose that C is compact as a subset of Y. Let $(W_i)_{i\in I}$ be a family of open subsets of X that covers C. By the definition of the subspace topology, $Y \cap W_i$ is an open subset of Y, and since $C \subseteq Y$ and $C \subseteq \bigcup_{i\in I} W_i$ it follows that

$$C \subseteq Y \cap \bigcup_{i\in I} W_i = \bigcup_{i\in I} (Y \cap W_i).$$

Thus $(Y \cap W_i)_{i\in I}$ is an covering of C by open subsets of Y, and since C is compact as a subset of Y it follows that there is a finite subset J of I such that $C \subseteq \bigcup_{i\in Y} (Y \cap W_i)$.

And since $Y \cap W_i \subseteq W_i$ for each i we deduce that $C \subseteq \bigcup_{i \in Y} W_i$, so that $(W_i)_{i \in J}$ is a finite subcovering of the original open covering of C. Hence C is compact as a subset of X, as required. □

As a corollary of the above two results we obtain a generalization of the familiar result that a continuous real valued function on a closed interval in $\mathbb{R}$ has a maximum value.

Corollary. *Let f be a continuous real valued function on a nonempty compact subset C of a topological space X. Then f has a maximum on C; that is, there exists $c \in C$ such that $f(c) \geq f(x)$ for all $x \in C$.*

Proof. By the preceding proposition, $f(C)$ is a nonempty compact subset of $\mathbb{R}$, and so by the Heine-Borel theorem it is closed and bounded. Put $M = \sup f(C)$. Since $f(C)$ is closed, $M \in f(C)$. (If M were in the open set $\mathbb{R} \setminus f(C)$ then there would be some interval $(M - \varepsilon, M + \varepsilon) \subseteq \mathbb{R} \setminus f(C)$, and then $M - \varepsilon$ would be an upper bound for $f(C)$ less than M, which was defined as the least upper bound.) Since $M \in f(C)$ there is a $c \in C$ with $f(c) = M$, and we see that $f(c) = \sup f(C) \geq f(x)$ for all $x \in C$, as required. □

Connectedness

Definition. A topological space X is *connected* if it is not the union of two nonempty disjoint open sets.

Since connectedness is defined just in terms of open sets, it is an example of a "topological property": a property which, if possessed by a topological space X, must be possessed also by any topological space homeomorphic to X. Topological spaces X and Y are homeomorphic if and only if there is a bijective correspondence $X \leftrightarrow Y$ that preserves open sets, in the sense that each subset of X is open if and only if the corresponding subset of Y is open. (This is equivalent to saying that the function from X to Y and its inverse from Y to X are both continuous.) Suppose that such a correspondence exists. Then if U, V are any two subsets of X, and U', V' the corresponding subsets of Y, then $X = U \cup V$ if and only if $Y = U' \cup V'$, and $U \cap V$ is empty if and only if $U' \cap V'$ is empty; furthermore, U and V are open if and only if U' and V' are open. So X is disconnected if and only if Y is disconnected.

The fact that homeomorphisms preserve connectedness can often be used to help prove that spaces are not homeomorphic. There are examples of such proofs in the tutorial exercises.

A subset A of a topological space X is said to be connected if it is a connected space. (That is, connected in the subspace topology.) Recall that the subsets of A that are open in the subspace topology are those of the form $A \cap U$, where U is open in X. So A is not connected if there exist U_1, U_2, open sets in X, such that $A \cap U_1$ and $A \cap U_2$ are both nonempty, $A = (A \cap U_1) \cup (A \cap U_2)$, and $(A \cap U_1) \cap (A \cap U_2) = \emptyset$. This simplifies marginally: A is not connected if and only if there exist open sets U_1 and U_2 such that $A \cap U_1$ and $A \cap U_2$ are both nonempty, $A \subseteq U_1 \cap U_2$ and $A \cap (U_1 \cap U_2) = \emptyset$.

The following simple way of characterizing disconnected sets, sometimes enables proofs to be shortened (albeit marginally). It links connectedness with continuity.

Proposition. *A topological space X is disconnected if and only if there is a continuous surjective function from X to the two-element discrete topological space.*

Proof. Let $S = \{0, 1\}$ be equipped with the discrete topology, so that $\{0\}$ and $\{1\}$ are both open sets. If X is disconnected then there exist disjoint nonempty open sets

U_1, $U_2 \subseteq X$ with $X = U_1 \cup U_2$, and so we may define a function $f\colon X \to \{0,1\}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in U_1, \\ 1 & \text{if } x \in U_2. \end{cases}$$

Then f is surjective since U_1 and U_2 are nonempty. To show that f is continous we must show that the preimages of all open subsets of $\{0,1\}$ are open, and this is trivial since $\{0,1\}$ has only four subsets altogether. Obviously $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(\{0,1\}) = X$ are both open, and, by the way that f was defined, $f^{-1}(\{0\}) = U_1$ and $f^{-1}(\{1\}) = U_2$ are also both open.

Conversely, suppose that there exists a continuous surjective function $f\colon X \to \{0,1\}$. Define $U_1 = f^{-1}(\{0\})$ and $U_2 = f^{-1}(\{1\})$.† Then U_1 and U_2 are open, since they are continuous preimages of open sets, disjoint because there can be no $x \in X$ such that $f(x)$ is both 0 and 1, and nonempty since f is surjective. Finally, if $x \in X$ is arbitrary then either $f(x) = 0$, giving $x \in U_1$, or $f(x) = 1$, giving $x \in U_2$. So $X = U_1 \cup U_2$, and so X is disconnected. □

† Some authors, including (regrettably) myself, sometimes write $f^{-1}(y)$ rather than $f^{-1}(\{y\})$ for the set $\{\, x \mid f(x) = y \,\}$. This practice is not to be recommended, since if f is bijective, so that an inverse function exists, then $f^{-1}(y)$ should be an element of X rather than a single element subset of X.