The Contraction Mapping Theorem

Definition. Let (X, d) be a metric space. A function $f\colon X \to X$ is called a *contraction mapping* if there exists a constant K with $0 \le K < 1$ such that $d(f(x), f(y)) \le Kd(x, y)$ for all x, $y \in X$.

Our chief aim today is to prove the following important (but not difficult) result, known as *Banach's Fixed Point Theorem*, or *Contraction Mapping Theorem*. It says that a contraction mapping on a complete metric space has a unique fixed point.

We need two easy lemmas.

Lemma. *Let $K \ge 0$, and suppose that $f\colon X \to X$ satisfies $d(f(x), f(y)) \le Kd(x, y)$ for all x, $y \in X$, where (X, d) is a metric space. Then f is uniformly continuous on X.*

Proof. Note that if $K = 0$ then $f(x) = f(y)$ for all x, $y \in X$, and it is trivial that f is uniformly continuous. Otherwise, given $\varepsilon > 0$, put $\delta = \varepsilon/K$. Then for all x, $y \in X$, if $d(x, y) < \delta$ then $d(f(x), f(y)) \le Kd(x, y) < K\delta = \varepsilon$, as required. □

The following elementary result should be familiar from junior or intermediate level mathematics courses, at least in the case of functions from $\mathbb{R}$ to $\mathbb{R}$.

Lemma. *Let X, Y be topological spaces, and $f\colon X \to Y$ a continuous function. If (x_n) is a sequence in X that converges to some point $x \in X$, then the sequence $(f(x_n))$ in Y converges to $f(x)$.*

Proof. Let $U \subseteq Y$ be any open neighbourhood of $f(x)$. Since f is continuous, $f^{-1}(U)$ is open in X. Since $x \in f^{-1}(U)$ and $x_n \to x$ as $n \to \infty$, there exists an N such that $x_n \in f^{-1}(U)$ for all $n > N$. Thus $f(x_n) \in U$ for all $n > N$. Since U was an arbitrary open neighbourhood of $f(x)$ this shows that $f(x_n) \to f(x)$ as $n \to \infty$, as required. □

We now come to Banach's theorem.

Theorem. *Let (X, d) be a complete metric space, with $X \ne \emptyset$, and let $f\colon X \to X$ be a contraction mapping. Then there is a unique point $t \in X$ such that $f(t) = t$.*

Proof. Since f is a contraction mapping there exists a nonnegative $K < 1$ such that $d(f(x), f(y)) \le Kd(x, y)$ for all x, $y \in X$. Fix such a K.

Choose a point $x_0 \in X$, and, proceeding recursively, for all $i \in \mathbb{Z}^+$ define x_i by $x_i = f(x_{i-1})$. Our first task is to show that $(x_i)_{i=0}^{\infty}$ is a Cauchy sequence. Now for each $n \in \mathbb{Z}^+$ we have $x_n = f(x_{n-1})$ and $x_{n+1} = f(x_n)$; so

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \le Kd(x_{n-1}, x_n).$$

Thus it follows that for all $n \in \mathbb{Z}^+$,

$$d(x_n, x_{n+1}) \le Kd(x_{n-1}, x_n) \le K^2 d(x_{n-2}, x_{n-1}) \le \cdots \le K^n d(x_0, x_1).$$

So if n, $m \in \mathbb{Z}^+$ with $m > n$ then, using the triangle inequality,

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\le (K^n + K^{n+1} + \cdots + K^{n+m-1}) d(x_0, x_1) \\ &\le \big(K^n \sum_{i=0}^{\infty} K^i\big) d(x_0, x_1) \\ &= \frac{K^n d(x_0, x_1)}{1 - K} \qquad \text{(since } K < 1\text{).} \end{aligned}$$

But $\lim_{n\to\infty} \frac{K^n d(x_0,x_1)}{1-K} = 0$ (since $K < 1$); so given $\varepsilon > 0$ we may choose an N such that $\frac{K^n d(x_0,x_1)}{1-K} < \varepsilon$ whenever $n > N$. It then follows that $d(x_n, x_m) < \varepsilon$ whenever $m > n > N$. By symmetry the same holds whenever $n > m > N$, and of course it is trivially true also when $n = m$. So we have shown that for all $\varepsilon > 0$ there is an N such that $d(x_n, x_m) < \varepsilon$ whenever n, $m > N$. Thus (x_i) is a Cauchy sequence.

Since X is complete and (x_i) is a the Cauchy sequence, there is a $t \in X$ such that $\lim_{i\to\infty} x_i = t$. Now by our first lemma above the function f is continuous, and hence by the other lemma it follows that $\lim_{i\to\infty} f(x_i) = f(t)$. But $f(x_i) = x_{i+1}$; so $\lim_{i\to\infty} x_{i+1} = f(t)$. But obviously $\lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} x_i = t$, and so we have shown that $f(t) = t$. Thus f has a fixed point in X.

Suppose that $u \in X$ is also a fixed point of f. Then $u = f(u)$ and $t = f(t)$; so

$$d(u,t) = d(f(u), f(t) \leq K d(u,t)$$

(since f is a contraction mapping). But $K < 1$, and if $d(u,t) > 0$ then it follows that $Kd(u,t) < d(u,t)$, a contradiction. So $d(u,t) \leq 0$, which forces $u = t$. Thus t is the unique fixed point of f. □

The above proof provides a good practical algorithm for finding the fixed point of a contraction mapping, to any desired degree of accuracy, provided only that values of the mapping can be computed efficiently. We choose an arbitrary point x_0 as an initial approximation to the fixed point t of the given function f, and then simply iterate f to find successive approximations x_n. Thus $x_n = f^{(n)}(x_0)$, where

$$f^{(n)} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ terms}}.$$

Now since $f(t) = t$ it follows that $f^{(n)}(t) = t$ for all n, and so

$$\begin{aligned} d(t, x_n) = d(f^{(n)}(t), f^{(n)}(x_0)) &\leq Kd(f^{(n-1)}(t), f^{(n-1)}(x_0)) \\ &\leq K^2 d(f^{(n-2)}(t), f^{(n-2)}(t)) \leq \cdots \leq K^n d(t, x_0), \end{aligned}$$

and this approaches 0 rapidly as $n \to \infty$. (Of course, the smaller the value of K the faster the convergence, but remember that, for example, K^n approaches 0 faster than $(1/n)^k$, no matter how large k is.)

A second point to note is that it is possible for a function f which is not contraction mapping to have the property that $f^{(n)}$ is a contraction mapping for some n. (There are examples of this in the tutorial and assignment exercises, and another below.) Then $f^{(n)}$ will have a unique fixed point. But it is easily shown that if f is such that $f^{(n)}$ has a unique fixed point, then that point is also a fixed point, and the unique fixed point, of the function f itself. To see this, suppose that t is the unique fixed point of $f^{(n)}$. Then

$$f^{(n)}(f(t)) = f^{(n+1)}(t) = f(f^{(n)}(t)) = f(t),$$

since $f^{(n)}(t) = t$, and thus we have shown that $f(t)$ is a fixed point of $f^{(n)}$. Since t is the unique fixed point of $f^{(n)}$, it follows that $f(t) = t$. So t is a fixed point of f. On the other hand, it is clear that any fixed point of f is also a fixed point of $f^{(n)}$, and since the fixed point of $f^{(n)}$ is unique the same must be true of f.

Two Examples

Example 1. Here is an easy example of a function f such that $f^{(2)}$ is a contraction mapping, but f is not. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} -2x & \text{if } x \geq 0, \\ -x/4 & \text{if } x < 0. \end{cases}$$

It is clear that if both x and y are positive then $d(f(x), f(y)) = 2d(x, y)$, and so f is not a contraction mapping. But since $f^{(2)}(x) = x/2$ for all x, it is also clear that $f^{(2)}$ is.

Example 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2} + x - \frac{x^2}{4}$. This function achieves a maximum value at $x = 2$, and its restriction to $[1, 2]$ is increasing. Since $f(1) = 1\frac{1}{4}$ and $f(2) = 1\frac{1}{2}$ it is certainly true that $f(x) \in [1, 2]$ for all $x \in [1, 2]$, and so there is a function $g\colon [1, 2] \to [1, 2]$ defined by $g(x) = f(x)$ for all $x \in [1, 2]$.

For all x, $y \in [1, 2]$ we have

$$g(x) - g(x) = x - y - \frac{x^2}{4} + \frac{y^2}{4} = (x - y)(1 - \frac{x}{4} - \frac{y}{4}),$$

and so

$$|g(x) - g(y)| \leq |x - y| \sup_{x,\, y \in [1,2]} \left| 1 - \frac{x}{4} - \frac{y}{4} \right| = \frac{1}{2}|x - y|,$$

since, for x, $y \in [1, 2]$, the maximum value that $1 - \frac{x}{4} - \frac{y}{4}$ can take clearly occurs when $x = y = 1$ and the minimum when $x = y = 2$, whence $0 \leq 1 - \frac{x}{4} - \frac{y}{4} \leq \frac{1}{2}$. So, with the metric d defined as usual for subsets of $\mathbb{R}$, we have $d(g(x), g(y) \leq \frac{1}{2} d(x, y)$, whence g is a contraction.

Since $[1, 2]$ is a closed subset of the complete space $\mathbb{R}$, it is itself a complete space (by the theorem proved at the start of Lecture 12). So the Contraction Mapping Theorem tells us that g has a fixed point in $[1, 2]$. Of course, it is easy in this case to directly solve the equation $g(x) = x$, since it is merely a quadratic. But similar examples involving polynomials of higher degree result in equations that cannot be solved algebraically, and then topological methods, such as the Contraction Mapping Theorem, are required. See the Tutorial exercises for examples.

Note also that, in the above example, if x is rational then so is $g(x)$. Hence g also yields a contraction mapping on the space $[1, 2] \cap \mathbb{Q}$. The Contraction Mapping Theorem does not tells that there is a fixed point in $[1, 2] \cap \mathbb{Q}$, because $[1, 2] \cap \mathbb{Q}$, unlike $[1, 2]$, is not closed as a subspace of $\mathbb{R}$, and hence it is not complete. And indeed there is no fixed point in $[1, 2] \cap \mathbb{Q}$, since the unique fixed point of g is in fact $\sqrt{2}$, which is not rational.