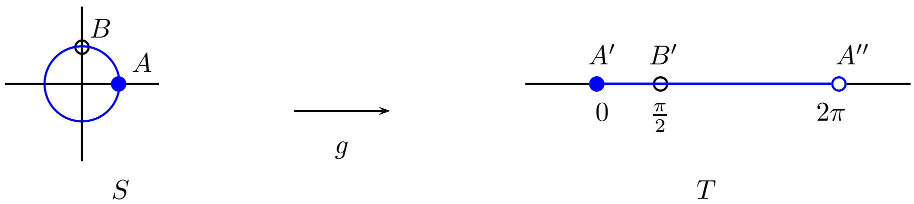
Homeomorphisms

At the end of last lecture an example was given of a bijective continuous function f such that f^{-1} is not continuous. For another example, consider the sets $T = [0, 2\pi) \subseteq \mathbb{R}$ and $S = \{\, (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \,\}$, and let $f\colon T \to S$ be given by $f(\theta) = (\cos\theta, \sin\theta)$. It is trivial that f is bijective. Viewed geometrically, f wraps the interval T around the circle S, joining the ends together. We consider T as a subspace of $\mathbb{R}$ and S as a subspace of $\mathbb{R}^2$, using the usual (Euclidean) metrics on $\mathbb{R}$ and $\mathbb{R}^2$, so that the concept of continuity for functions between S and T is the familiar concept studied in calculus courses. The function f is thus clearly continuous, since its component functions, cos and sin, are differentiable and therefore continuous. Joining the ends of the interval together is thus perfectly compatible with continuity. However, the inverse function $g\colon S \to T$ cuts the circle open at the point $(1, 0)$ and unwraps it. This is not a continuous operation: some points that are close together in S are mapped by g to points that are far apart in T.



In the diagram, A is the point $(1, 0)$ in S, and g maps it to the point A' in T. But points arbitrarily near to A that are below the x-axis are mapped to points in T that are near to A'' rather than A'. This tells us that g is discontinuous at A. Indeed, it wiil be possible to find an open neighbourhood of $A' = g(A)$ in T whose preimage in S is not open. A suitable such open neighbourhood of A' is $T \cap (-\pi/2, \pi/2) = [0, \pi/2)$ (which, although it is not open in $\mathbb{R}$, is open in T, since it is the intersection of T with an open subset of R). The preimage in S of this neighbourhood is the arc from A to B, including A but not B. This is not an open subset of S, since it contains A but no neighbourhood of A.†

Of course, it is possible for the inverse of a continuous bijective function to be continous, and indeed functions of this kind are very important in topology.

Definition. Let X and Y be topological spaces. A *homeomorphism* from X to Y is a bijective function $f\colon X \to Y$ such that f and f^{-1} are both continuous. If there is a homeomorphism $X \to Y$ we say that X is *homeomorphic* to Y, and we write $X \cong Y$.

It is clear that $\cong$ is an equivalence relation on topological spaces. Note that if $f\colon X \to Y$ is a bijective function and $g\colon Y \to X$ its inverse, then $S \mapsto f(S) = g^{-1}(S)$ is a bijective function from the set of all subsets of X to the set of all subsets of Y. The inverse function from the subsets of Y to the subsets of X is given by $T \mapsto g(T) = f^{-1}(T)$.‡ When f and g are both continuous then these functions take open sets to open sets. Thus homeomorphic topological spaces have the same topological structure: they can be

† See also the solutions to Tutorial 7, Exercise 1.

‡ Remember, however, that $S \mapsto f(S)$ and $T \mapsto f^{-1}(T)$ are not inverses of each other when f is not bijective. In general all one can say is that $S \subseteq f^{-1}(f(S))$ and $f(f^{-1}(T)) \subseteq T$.

regarded as different incarnations of the same abstract space, the homeomorphism being simply a relabelling of the points.

If (X, d_X) and (Y, d_Y) are metric spaces that are homeomorphic topological spaces then we also say that X and Y are *topologically equivalent.*

In the example considered atthe end of Lecture 16, the function $f\colon [0,1]\cup(2,3] \to [0,2]$ is not a homeomorphism, since its inverse is not continuous. However, this does not show that the spaces $[0,1] \cup (2,3]$ and $[0,2]$ are not homeomorphic: it is conceivable that there might be some other function $[0,1] \cup (2,3] \to [0,2]$ which is a homeomorphism. In fact there is not—it is true that these spaces are not homeomorphic—but to prove this we need to find some topological property possessed by one of the spaces but not the other. Here by "topological property" we mean something that can be defined just in terms of open sets. If the spaces were homeomorphic there would be no such property, since a bijective correspondence that preserves open sets would also preserve properties defined in terms of open sets.

A suitable property in this instance is connectedness, of which we shall have more to say later. A space is said to be connected if it cannot be expressed as the disjoint union of two nonempty open sets. Equivalently, X is connected if and only if X and $\emptyset$ are the only subsets of X that are both open and closed. It is clear that $X = [0,1] \cup (2,3]$ is disconnected, since $[0,1] = X \cap (-1/2, 3/2)$ and $(2,3] = X \cap (3/2, 7/2)$ are both open subsets of X. On the other hand, $Y = [0,2]$ is a connected space, since in fact all intervals in $\mathbb{R}$ are connected (as we shall prove later). So Y is not homeomorphic to X.

The following theorem lists a number of characterizations of homeomorphisms. These all follow readily from things we have proved previously.

Theorem. *Let X and Y be topological spaces, and let $f\colon X \to Y$ be a bijective function. The following conditions are equivalent:*

(i) f is a homeomorphism;
(ii) for all $U \subseteq X$, the set U is open in X if and only if $f(U)$ is open in Y;
(iii) for all $F \subseteq X$, the set F is closed in X if and only if $f(F)$ is closed in Y;
(iv) $f(\overline{S}) = \overline{f(S)}$, for all $S \subseteq X$;
(v) $f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$, for all $T \subseteq Y$;
(vi) $f(\mathrm{Int}(S)) = \mathrm{Int}(f(S))$, for all $S \subseteq X$;
(vii) $f^{-1}(\mathrm{Int}(T)) = \mathrm{Int}(f^{-1}(T))$, for all $T \subseteq Y$.

Additionally, if X and Y are metric spaces then the following condition is also equivalent to the conditions above:

(viii) for all sequences $(x_n)_{n=0}^{\infty}$ in X and all $x \in X$, the sequence $(f(x_n))_{n=0}^{\infty}$ in Y converges to $f(x)$ if and only if (x_n) converges to x.

Proof. We proved above that (i) implies (ii): if f is a homeomorphism then $U = f^{(-1)}(V)$ is open in X whenever V is open in Y (since f is continuous), and $V = (f^{(-1)})^{(-1)}(U)$ is open in Y whenever U is open in X (since $f^{(-1)}$ is continuous). Conversely, if (ii) holds then both f and $f^{(-1)}$ have the property that preimages of open sets are open; so f and $f^{(-1)}$ are both continuous, and (i) holds.

The equivalence of (ii) and (iii) follows readily from the fact that a set is closed if and only if its complement is open, and if F is the complement of U then $f(F)$ is the complement of $f(U)$ (given that f is bijective). The proofs that (iv), (v), (vi) and (vii) are all equivalent to (i), (ii) and (iii) are similarly easy, and are left as exercises for the reader. (See also the solutions to Exercises 7 and 8 of Tutorial 6.)

If X and Y are metric spaces then by Question 1 of Assignment 2 we know that f is continuous at the point $x \in X$ if and only if for all sequences $(x_n)_{n=1}^{\infty}$ in X, if $x_n \to x$ as $n \to \infty$, then $f(x_n) \to f(x)$ as $n \to \infty$. Applying the same result to f^{-1} we see that f^{-1} is continuous at $f(x)$ if and only if for all sequences $(x_n)_{n=1}^{\infty}$ in X, if $f(x_n) \to f(x)$ as $n \to \infty$ then $x_n \to x$ as $n \to \infty$. So condition (viii) is equivalent to continuity of both f and f^{-1}; that is, (viii) is equivalent to (i). □

Equivalent metrics

If d_1 and d_2 are metrics on the same set X then we say that d_1 and d_2 are *equivalent metrics* if the collection of subsets of X that are open relative to d_1 is the same as the collection of subsets of X that are open relative to d_2. Another way to express this condition is to say that the identity mapping from X to itself, considered as a mapping from the metric space (X, d_1) to the metric space (X, d_2), is a homeomorphism.

By the epsilon-delta characterization of continuity (see below), the identity mapping $(X, d_1) \to (X, d_2)$ is continuous if and only if for all $x \in X$ and all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t \in X$, if $d_1(x, t) < \delta$ then $d_2(x, t) < \varepsilon$. That is, every open ball $B_{d_2}(x, \varepsilon)$ contains some open ball $B_{d_1}(x, \delta)$. Expressed a little less formally, this says that you can ensure that t is d_2-close to x by making it d_1-close to x. The metrics are equivalent if and only if the identity mappings $(X, d_1) \to (X, d_2)$ and $(X, d_2) \to (X, d_1)$ are both continuous, and this is equivalent to saying that for all $x \in X$, every open ball of (X, d_2) with centre x contains an open ball of (X, d_1) with centre x, and every open ball of (X, d_1) with centre x contains an open ball of (X, d_2) with centre x.

Equivalence of metrics can also be characterized in terms of convergence of sequences. If d_1 and d_2 are equivalent and $(x_n)_{n=1}^{\infty}$ is a sequence in X that converges, as a sequence in (X, d_1), to some point $x \in X$, then it is easy to show that as a sequence in (X, d_2) it must still converge to x. For, given any $\varepsilon > 0$, one may choose an $\varepsilon' > 0$ such that $B_{d_1}(x, \varepsilon') \subseteq B_{d_2}(x, \varepsilon)$, and, because $\lim\limits_{n\to\infty} x_n = x$ in (X, d_1), there exists an N such that $x_n \in B_{d_1}(x, \varepsilon')$ for all $n > N$. It follows that $x_n \in B_{d_2}(x, \varepsilon)$ for all $n > N$, and because ε was arbitrary this shows that $\lim\limits_{n\to\infty} x_n = x$ in (X, d_2). Symmetrically, any convergent sequence in (X, d_2) will also converge, to the same limit, when considered as a sequence in (X, d_1).

Conversely, if every convergent sequence in either space is also convergent in the other space, with the same limit, then it follows that the metrics are equivalent. For suppose that the condition holds, and let $F \subseteq X$ be any set that is closed relative to d_1. Let $\overline{F}$ be the closure of F relative to d_2. By a proposition we proved in Lecture 8, there is a sequence (x_n) of points of F such that $x_n \to x$ as $n \to \infty$, in (X, d_2). By the hypothesis, $x_n \to x$ as $n \to \infty$ also in (X, d_1). But F is closed in (X, d_1), and $x_n \in F$ for all n; so $\lim\limits_{n\to\infty} x_n \in F$. So $x \in F$, and since x was an arbitrary element of $\overline{F}$ we have shown that $\overline{F} \subseteq F$ (and thus $\overline{F} = F$). So F is closed in (X, d_2). Symmetrically, every closed set in (X, d_2) is also closed in X_{d_1}. And since the open sets are just the complements of the closed sets, it follows that the open sets of (X, d_1) are the same as the open sets of (X, d_2). So the metrics are equivalent, as claimed.

Uniform continuity

Let (X, d_X) and (Y, d_Y) be metric spaces, and $f\colon X \to Y$ a function. The epsilon-delta characterization of continuity is as follows: f is continuous on X if and only if for all $x \in X$ and all $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all $t \in X$, if $d_X(x, t) < \delta$ then

$d_Y(f(x), f(t)) < \varepsilon$.

Definition. The function f is said to be *uniformly continuous* on X if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all x, $t \in X$, if $d_X(x,t) < \delta$ then $d_Y(f(x), f(t)) < \varepsilon$.

The difference between continuity and uniform continuity is that for continuity the number δ is allowed to depend on x as well as on ε, whereas for uniform continuity δ must depend on ε only. It is clear, therefore, that uniform continuity implies continuity.

Our next result gives a sufficient but not necessary condition for metrics d_1 and d_2 to be equivalent.

Proposition. *Let d_1 and d_2 be metrics on the set X. If there exist k_1, $k_2 > 0$ such that*

$$k_1 d_1(x,y) < d_2(x,y) < k_2 d_1(x,y) \qquad \text{for all } x,\ y \in X,$$

then d_1 and d_2 are equivalent.

Proof. We show that the identity mapping $f\colon (X, d_1) \to (X, d_2)$ is uniformly continuous. Indeed, given $\varepsilon > 0$ we can define $\delta = k_2^{-1}\varepsilon$, and then for all x, $t \in X$, if $d_1(x,t) < \varepsilon$ then

$$d_2(f(x), f(y)) = d_2(x,y) \le k_2 d_1(x,y) < k_2\delta = \epsilon,$$

as required. Similarly, the identity mapping $g\colon (X, d_2) \to (X, d_1)$ is uniformly continuous, since given $\varepsilon > 0$ we can define $\delta = k_1\varepsilon$, and then for all x, $t \in X$, if $d_2(x,t) < \varepsilon$ then

$$d_1(g(x), g(y)) = d_1(x,y) \le k_1^{-1} d_1(x,y) < k_1^{-1}\delta = \epsilon.$$

Since the identity functions in both directions are continuous and inverse to each other, they are homeomorphisms. Hence the metrics are equivalent. □

Additional examples

To illustrate the ideas we have been dealing with, we provide here some extra examples that were not presented in the lecture.

Example 1. Let $p \ge 1$ and let d_p be the metric on $\mathbb{R}^n$ defined by

$$d_p(x,y) = \sqrt[p]{|x_1 - y_1|^p + |x_2 - y_2|^p + \cdots + |x_n - y_n|^p}.$$

Also, let d_∞ be the sup metric,

$$d_\infty(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|).$$

Then d_p and d_∞ are equivalent. (We mentioned this fact previously, in Lecture 7.) The proof consists in showing that

$$d_\infty(x,y) \le d_p(x,y) \le n^{1/p} d_\infty(x,y).$$

The details are trivial, and are left to the reader. (See also the solution of Question 5 of Tutorial 7.)

Example 2. Let $\mathcal{C}$ be the set of all continuous functions on $[0,1]$, and let d_1, d_∞ be the metrics on $\mathcal{C}$ defined by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx$$

$$d_\infty(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

for all $f,\, g \in \mathcal{C}$. Then d_1 and d_∞ are not equivalent.

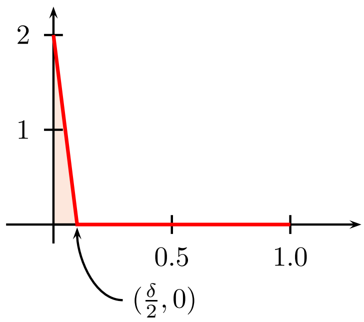
To prove this we show that if f is the constant function $f(x) = 0$ for all $x \in [0,1]$ then the open ball $B_{d_\infty}(f,1)$ does not contain any open ball $B_{d_1}(f,\delta)$ (where $\delta > 0$). Note first that

$$B_{d_\infty}(f,1) = \{\, g \in \mathcal{C} \mid d_\infty(f,g) < 1 \,\} = \{\, g \in \mathcal{C} \mid \sup_{x\in[0,1]} |g(x)| < 1 \,\},$$

whereas

$$B_{d_1}(f,\delta) = \{\, g \in \mathcal{C} \mid d_1(f,g) < \delta \,\} = \{\, g \in \mathcal{C} \mid \int_0^1 |g(x)|\, dx < \delta \,\},$$

Our task is to show that no matter how small δ is (provided it is positive), there exists a function $g \in B_{d_1}(f,\delta)$ that is not in $B_{d_\infty}(f,1)$. So it suffices to produce a g with $\int_0^1 |g(x)|\, dx < \delta$ and $g(t) \geq 1$ for some $t \in [0,1]$. This is easily achieved, since we can make the area under the graph of $|g(x)|$ as small as we please by restricting the region on which $|g(x)| \geq 1$ to an extremely small subinterval of $[0,1]$. So let us choose g to have graph as shown in the diagram.



Thus, g is defined by the following formula:

$$g(x) = \begin{cases} 2 - \frac{4}{\delta}x & \text{if } 0 \leq x \leq \delta/2, \\ 0 & \text{if } \delta/2 < x \leq 1. \end{cases}$$

The value of $\int_0^1 |g(x)|\, dx$ is the area of the shaded triangular region in the diagram, which is $\delta/2$, since the base has length $\delta/2$ and the height is 2. So $g \in B_{d_1}(f,1)$. And since $g(0) > 1$ it follows that $g \notin B_{d_\infty}(f,1)$.

Note that these calculations have shown us that the identity function, from $\mathcal{C}$ with metric d_1 to $\mathcal{C}$ with metric d_∞, is not continuous. One cannot ensure that g will be close to f in $(\mathcal{C}, d_\infty)$ by only insisting that g be chosen close to f in $(\mathcal{C}, d_1)$. However, the identity mapping in the other direction, $(\mathcal{C}, d_\infty) \to (\mathcal{C}, d_1)$, is continuous, since

$$\begin{aligned} d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx &\leq \int_0^1 \sup_{t\in[0,1]} |f(t) - g(t)|\, dx \\ &= (1-0) \sup_{t\in[0,1]} |f(t) - g(t)| = d_\infty(f,g) \end{aligned}$$

for all $f,\, g \in \mathcal{C}$. We can ensure that f and g are close in $(\mathcal{C}, d_1)$ by requiring them to be close in $(\mathcal{C}, d_\infty)$.

Our two examples have shown that for all $p \geq 1$, including $p = \infty$, the metrics d_p on $\mathbb{R}^n$ are all equivalent, but that the analogous metrics on $\mathcal{C}$ are not equivalent. It is natural to also ask about the space X of all infinite sequences $a = (a_i)_{i=1}^{\infty}$ of real numbers such that $\sum_{i=1}^{\infty} |a_i|$ converges. For each $p \geq 1$ we may define a metric d_p on this space by $d_p(a,b) = \left(\sum_{i=1}^{\infty} |a_i - b_i|^p\right)^{1/p}$. (Note that one may use the comparison test to deduce the convergence of $\sum_{i=1}^{\infty} |a_i - b_i|^p$ from the convergence of $\sum_{i=1}^{\infty} |a_i - b_i|$ whenever $p \geq 1$.) So are the metrics d_p for different values of p equivalent?

The answer to this question is no. This should not be surprising, since if $q > p$ then convergence of $\sum_{i=1}^{\infty} |a_i|^q$ does not guarantee convergence of $\sum_{i=1}^{\infty} |a_i|^p$. If we let z be the zero sequence, $z_i = 0$ for all i, then $B_{d_q}(z, \delta)$ consists of all $a \in X$ such that $\sum_{i=1}^{\infty} |a_i|^q < \delta$. We shall show that no matter how small δ is, we can find an a satisfying this condition such that $a \notin B_{d_p}(z, 1)$. Indeed, if $x_i = \sqrt[p]{1/i}$ then we can show, using the integral test, that $\sum_{i=1}^{\infty} x_i^q$ converges (given that $q > p$). But $\sum_{i=1}^{\infty} x_i^p$ is the harmonic series, well known to to diverge. If we put $S = \sum_{i=1}^{\infty} x_i^q$, and $y_i = x_i(\delta/S)^{1/q}$, then $\sum_{i=1}^{\infty} y_i^q = \delta$. But we can choose $k \in \mathbb{Z}^+$ large enough so that $\sum_{i=1}^{k} y_i^p = (\delta/S)^{p/q} \sum_{i=1}^{k} (1/i) > 1$, and we can obtain an element $a \in X$ by setting $a_i = y_i$ for $i \leq k$ and $a_i = 0$ for $i > k$. Then

$$d_q(a,z) = \sum_{i=1}^{\infty} |a_i - z_i|^q = \sum_{i=1}^{\infty} |a_i|^q = \sum_{i=1}^{k} |a_i|^q = \sum_{i=1}^{k} y_i^q < \sum_{i=1}^{\infty} y_i^q = \delta,$$

so that $a \in B_{d_q}(z, \delta)$, but

$$d_p(a,z) = \sum_{i=1}^{\infty} |a_i - z_i|^p = \sum_{i=1}^{\infty} |a_i|^p = \sum_{i=1}^{k} |a_i|^p = \sum_{i=1}^{k} y_i^p > 1,$$

so that $a \notin B_{d_p}(z, 1)$.