Product topology

Recall that in the last lecture we defined the concept of a base for a topology: a collection $\mathcal{B}$ of open sets is called a base if every open set can be expressed as a union of sets in $\mathcal{B}$. It is natural to ask what conditions a collection of subsets of an arbitrary set X must satisfy in order to be a base for some topology on X. The next proposition provides the answer.

Proposition. *Let $\mathcal{B}$ be a collection of subsets of a set X. Then $\mathcal{B}$ is a base for a topology on X if and only if $X = \bigcup_{B\in\mathcal{B}} B$ and for all B_1, $B_2 \in \mathcal{B}$ the set $B_1 \cap B_2$ is a union of sets in $\mathcal{B}$. When this condition is satisfied, the topology determined by $\mathcal{B}$ consists of all subsets U of X that are expressible as unions of sets in $\mathcal{B}$. That is, U is open if and only if there is a subcollection $\mathcal{D}$ of $\mathcal{B}$ such that $U = \bigcup_{B\in\mathcal{D}} B$.*

Proof. Assume first that $\mathcal{B}$ is a base for a topology. Then the fact that X is open ensures that $X = \bigcup_{B\in\mathcal{B}} B$, and the fact that the intersection of two open sets is open ensures that $B_1 \cap B_2$ is a union of sets in $\mathcal{B}$ whenever B_1, $B_2 \in \mathcal{B}$. So $\mathcal{B}$ satisfies the two specified conditions.

Conversely, suppose that $\mathcal{B}$ satisfies the specified conditions, and define $\mathcal{U}$ to be the collection of all $U \subseteq X$ such that $U = \bigcup_{B\in\mathcal{D}} B$ for some subcollection $\mathcal{D}$ of $\mathcal{B}$. Taking the subcollection $\mathcal{D}$ to be empty shows that $\emptyset \in \mathcal{U}$, and taking $\mathcal{D} = \mathcal{B}$ shows that $X \in \mathcal{U}$. If $(U_i)_{i\in I}$ is a family of sets such that $U_i \in \mathcal{U}$ for each $i \in I$, then for each $i \in I$ there is a subset $\mathcal{D}_i$ of $\mathcal{B}$ such that $U_i = \bigcup_{B\in\mathcal{D}_i} B$, and since

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} \bigcup_{B\in\mathcal{D}_i} B = \bigcup_{B\in\mathcal{D}} B,$$

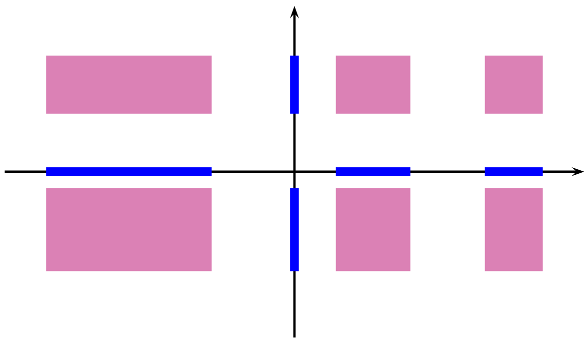
where $\mathcal{D} = \bigcup_{i\in I} D_i$, it follows that $\bigcup_{i\in I} U_i \in \mathcal{U}$. Finally, if U and V are arbitrary sets in $\mathcal{U}$ then $U = \bigcup_{B\in\mathcal{D}} B$ and $V = \bigcup_{C\in\mathcal{E}} C$ for some $\mathcal{D}$, $\mathcal{E} \subseteq \mathcal{B}$, and it follows that $U \cap V = \bigcup_{B\in\mathcal{D}} \bigcup_{C\in\mathcal{E}} B \cap C$ is a union of sets in $\mathcal{B}$, since each of the sets $B \cap C$ is a union of sets in $\mathcal{B}$. Thus $U \cap V \in \mathcal{U}$. □

We turn now to the question of how to make the Cartesian product of two topological spaces into a topological space. One's first guess might be that the open sets of $X \times Y$ should be all subsets of $X \times Y$ of the form $U \times V$, where U is an open subset of X and V an open subset of Y. However the union of a collection of sets of the form $U \times V$ is not necessarily also of the same form; this is demonstrated below in the case $X = Y = \mathbb{R}$. So in fact the appropriate way to define a topology on $X \times Y$ is to specify that collection

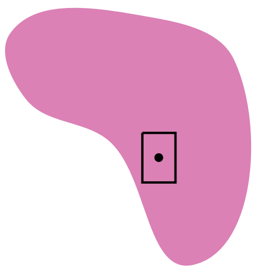
$$\mathcal{B} = \{\, U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y \,\}$$

is a base for the topology, rather than the whole topology.

The open subsets of $\mathbb{R}$ (with the usual topology) are those sets that are disjoint unions of open intervals; so any subset of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ that has the form $U \times V$ with U and V open in $\mathbb{R}$ will be a disjoint union of open rectangles (where an open rectangle is a set of the form $(a, b) \times (c, d) = \{\, (x, y) \mid a < x < b \text{ and } c < x < d \,\}$, where (a, b) and (c, d) are open intervals in $\mathbb{R}$). The first diagram below depicts $U \times V$ when U is a disjoint union of three intervals (identified with the subset of the X-axis marked in the diagram) and V a disjoint union of two intervals (identified with a subset of the Y-axis). Now it is easily seen that any subset $\mathbb{R}^2$ that is open in the topology derived from d, the Euclidean metric, can be expressed as a union of open rectangles. As with the proof that open sets are unions of open balls (see last lecture), to prove this it suffices to show that each point of a given open subset U of $\mathbb{R}^2$ lies in an open rectangle contained in U. Now if $(x, y) \in U$ then $B_d((x, y), \varepsilon) \subseteq U$ for some $\varepsilon > 0$, and if we put $\delta = \varepsilon/\sqrt{2}$ then it can be seen that $(x - \delta, x + \delta) \times (y - \delta, y + \delta) \subset U$. Thus sets which are expressible as unions of open rectangles need not be expressible as disjoint unions of open rectangles: there are open sets in $\mathbb{R}^2$ (such as circles) that do not have the form $U \times V$ for open subsets U and V of $\mathbb{R}$.†



The Cartesian product of $(-3, -1) \cup (0.5, 1.4) \cup (2.3, 3.0)$
and $(-1.2, -0.2) \cup (0.7, 1.4)$.



For any point x of an open set U in $\mathbb{R}^2$ one can find a
rectangle containing x and contained in U.
So U is the union of the rectangles it contains.

The following proposition is needed to justify the definition of the product topology foreshadowed above.

Proposition. *Let X and Y be topological spaces, and let $\mathcal{B}$ be the collection of all subsets of $X \times Y$ of the form $U \times V$ such that U is an open subset of X and V an open subset of Y. Then $\mathcal{B}$ is a base for a topology on $X \times Y$.*

† By contrast, in $\mathbb{R}$ any union of open intervals is also a disjoint union of open intervals. In the present context this should be regarded as anomolous behaviour: it is not usually the case that if $\mathcal{B}$ is a base for a topology on a set X then all open sets are disjoint unions of sets in $\mathcal{B}$.

Proof. Since X is an open subset of X and Y is an open subset of Y, it follows that the set $X \times Y$ itself is in the collection $\mathcal{B}$. Hence $X \times Y = \bigcup_{B \in \mathcal{B}} B$. By our previous proposition above, it remains to show that the intersection of any elements $B_1, B_2 \in \mathcal{B}$ is a union of elements of $\mathcal{B}$.

In fact it is easily seen that if $B_1, B_2 \in \mathcal{B}$ then $B_1 \cap B_2 \in \mathcal{B}$. To prove this, let U_1, U_2 be open subsets of X and V_1, V_2 open subsets of Y such that $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$. Then

$$\begin{aligned} B_1 \cap B_2 &= \{\, (x, y) \mid (x, y) \in U_1 \times V_1 \text{ and } (x, y) \in U_2 \times V_2 \,\} \\ &= \{\, (x, y) \mid x \in U_1,\ y \in V_1 \text{ and } x \in U_2,\ y \in V_2 \,\} \\ &= \{\, (x, y) \mid x \in U_1 \cap U_2 \text{ and } y \in V_1 \cap V_2 \,\} \\ &= (U_1 \cap U_2) \times (V_1 \cap V_2), \end{aligned}$$

and this is in the collection $\mathcal{B}$ since $U_1 \cap U_2$ is open in X (since U_1 and U_2 both are) and $V_1 \cap V_2$ is open in Y (since V_1 and V_2 both are). □

Definition. The topology on $X \times Y$ determined by the base $\mathcal{B}$ described in the above proposition is called the *product topology*.

Let X and Y be topological spaces, and suppose that a topology is defined on $X \times Y$ that is not necessarily the product topology. There are two obvious projection maps, π_X and π_Y, defined by

$$\begin{aligned} \pi_X \colon X \times Y &\to X \\ (x, y) &\mapsto x, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \pi_Y \colon X \times Y &\to Y \\ (x, y) &\mapsto y. \end{aligned}$$

It is natural to ask under what circumstances these mappings are continuous.

We know that π_X is continuous if and only if $\pi_X^{-1}(U)$ is open whenever U is open, and π_Y is continuous if and only if $\pi_Y^{-1}(V)$ is open whenever V is open. Now observe that if U is any open subset of X then

$$\pi_X^{-1}(U) = \{\, (x, y) \in X \times Y \mid \pi_X(x, y) \in U \,\} = \{\, (x, y) \in X \times Y \mid x \in U \,\} = U \times Y,$$

and similarly if V is any open subset of Y then $\pi_Y^{-1}(V) = X \times V$. So π_X and π_Y are both continuous if and only if $U \times Y$ and $V \times X$ are open subsets of $X \times Y$ for all open subsets U of X and V of Y. Since $(U \times Y) \cap (V \times X) = U \times V$, if $U \times Y$ and $X \times V$ are both open then $U \times V$ is open; conversely, if all subsets of $X \times Y$ of the form $U \times V$, with U open in X and V open in Y, are open in $X \times Y$, then, in particular, taking $V = Y$ we see that $U \times Y$ is open whenever U is open, and, similarly, taking $U = X$, we see that $X \times V$ is open whenever V is open.

We conclude from this that π_X and π_Y are both continuous precisely if $U \times V$ is open in $X \times Y$ whenever U is open in X and V is open in Y. Since these sets $U \times V$ form a base for the product topology, we see that the product topology on $X \times Y$ makes π_X and π_Y continuous. Furthermore, any other topology $\mathcal{T}$ on $X \times Y$ for which π_X and π_Y are both continuous must have the property that any subset of $X \times Y$ that is open in the product topology must be open in $\mathcal{T}$. The product topology is the *coarsest* (fewest open sets) such that the projections are continuous, every other topology with this property must be *finer* (more open sets).

Remarks

1. In future, whenever we deal with the Cartesian product of two topological spaces, unless explicitly stated otherwise, we shall regard the Cartesian product as a topological space via the product topology.

2. If (X, d_X) and (Y, d_Y) are metric spaces then we can make $X \times Y$ into a metric space by defining $d((x_1, y_1), (x_2, y_2)) = \max(d(x_1, x_2), d(y_1, y_2))$. With this definition, the open balls in $X \times Y$ are precisely the sets of the form $U \times V$ such that U is an open ball in X and V an open ball in Y, since for all $a \in X$, $b \in Y$ and $\varepsilon > 0$,

$$\begin{aligned} B_d((x, y), \varepsilon) &= \{\, (x, y) \mid d((a, b), (x, y)) < \varepsilon \,\} \\ &= \{\, (x, y) \mid d_X(a, x) < \varepsilon \text{ and } d_Y(b, y)) < \varepsilon \,\} \\ &= B_{d_X}(a, \varepsilon) \times B_{d_Y}(b, \varepsilon). \end{aligned}$$

Consequently the topology on $X \times Y$ determined by these open balls is precisely the product topology (where the topology on X is determined by the open balls in X and the topology on Y is determined by the open balls in Y).

Note that there are several other ways to define metrics on the Cartesian product. For example, for any $p \geq 1$ we could define

$$d((x_1, y_1), (x_2, y_2)) = \sqrt[p]{(d(x_1, x_2)^p + d(y_1, y_2)^p)};$$

furthermore, taking the limit as $p \to \infty$ gives back our previous definition. These alternatives are all topologically equivalent, in that they give rise to the same collections of open sets in $X \times Y$.‡

Theorem. *Let X, Y and Z be topological spaces, and let $f\colon Z \to X$ and $g\colon Z \to Y$ be functions. Define $f \times g\colon Z \to X \times Y$ by $(f \times g)(z) = (f(z), g(z))$ for all $z \in Z$. If f and g are both continuous then $f \times g$ is continuous.*

Proof. Suppose that f and g are continuous, and let O be an open set in $X \times Y$. Then O is a union $\bigcup_{i \in I}(U_i \times V_i)$ (for some indexing set I), where each U_i is open in X and each V_i open in Y. Now

$$\begin{aligned} (f \times g)^{-1}(O) &= \{\, z \in Z \mid (f \times g)(z) \in \bigcup_{i \in I}(U_i \times V_i) \,\} \\ &= \{\, z \in Z \mid (f \times g)(z) \in (U_i \times V_i) \text{ for some } i \in I \,\} \\ &= \bigcup_{i \in I}(f \times g)^{-1}(U_i \times V_i), \end{aligned}$$

and furthermore

$$\begin{aligned} (f \times g)^{-1}(U_i \times V_i) &= \{\, z \in Z \mid (f(z), g(z)) \in (U_i \times V_i) \,\} \\ &= \{\, z \in Z \mid f(z) \in U_i \text{ and } g(z) \in V_i \,\} \\ &= f^{-1}(U_i) \cap g^{-1}(V_i). \end{aligned}$$

This is an open set, for each i, since the intersection of two open sets is open, and the fact that f is continuous tells us that $f^{-1}(U_i)$ is open, and the fact that g is continuous tells

‡ Just as, for all $p \geq 1$, the metrics d_p on $\mathbb{R}^n$ all determine the same topology on $\mathbb{R}^n$—see Lecture 7.

us that $g^{-1}(V_i)$ is open. Thus $(f \times g)^{-1}(O)$ is a union of open sets, and therefore open. As this applies for all open subsets O of $X \times Y$, it follows that $f \times g$ is continuous. $\square$

The converse of the above result is also valid: if $f \times g$ is continuous then f and g are both continuous. The point is that $f = \pi_X \circ (f \times g)$, since for all $z \in Z$,

$$(\pi_X \circ (f \times g))(z) = \pi_X(f \times g)(z)) = \pi_X((f(z), g(z)) = f(z).$$

But the composite of two continuous functions is continuous; so since π_X is continuous, if $f \times g$ is also continous then it follows that f is continuous. A similar proof applies for g.

The theorem above makes it easy for us to determine if a function from $\mathbb{R}$ to $\mathbb{R}^n$ is continuous, since such functions are usually specified by giving their component functions. For example, the function $\mathbb{R} \to \mathbb{R}^3$ given by $x \mapsto (e^x, x^2+1, (\sin x - x)^2)$ is continuous, since $x \mapsto e^x$, $x \mapsto x^2+1$ and $x \mapsto (\sin x - x)^2$ are all continuous. (Strictly, to prove this we must make two applications of the theorem, and identify $\mathbb{R}^3$ with $\mathbb{R} \times (\mathbb{R} \times \mathbb{R}))$ or $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ in the obvious way.)

Homeomorphisms

A *homeomorphism* from one topological space to another is a bijective function f such that f and f^{-1} are both continuous. It is important to note that continuity of f does not guarantee continuity of f^{-1}; we give an example to demonstrate this before discussing homeomorphisms.

let d be the usual metric on $\mathbb{R}$ and d' the discrete metric (for which $d'(x,y) = 1$ whenever $x \neq y$). Observe that for all $x \in R$ the open ball $B_{d'}(x, 1/2)$ is just the singleton set $\{x\}$. Thus all singleton sets, and consequently all sets, are open with respect to the topology on $\mathbb{R}$ derived from d'. Let the topological space X be $\mathbb{R}$ equipped with this topology, and let Y be $\mathbb{R}$ equipped with the usual topology (derived from the metric d). Let $f: X \to Y$ be the identity function $\mathbb{R} \to \mathbb{R}$. Obviously f is bijective, its inverse $g: Y \to X$ being also the identity function. Furthermore, if U is any open subset of Y then $f^{-1}(U)$ is an open subset of X, since every subset of X is open. Thus f is continous. However, g is not continous, since $\{0\}$ is an open subset of X, but $g^{-1}(\{0\}) = \{0\}$ is not an open subset of Y.

We give also another example, this time without resorting to the use of the discrete topology. Let $X = [0,1] \cup (2,3]$ and $Y = [0,2]$, both regarded as metric subspaces of $\mathbb{R}$ with the usual metric. Since $Y = [0,1] \cup (1,2]$ it is easy to see that the function $f: X \to Y$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ x-1 & \text{if } 2 < x \le 3, \end{cases}$$

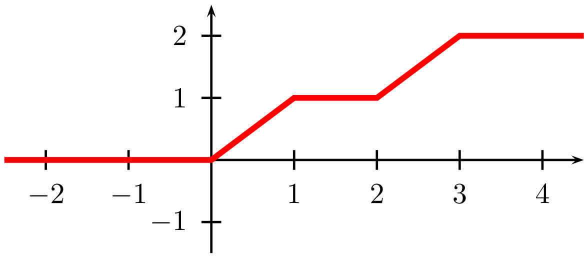
is bijective, its inverse $g: Y \to$X being given by

$$g(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ x+1 & \text{if } 1 < x \le 2. \end{cases}$$

We show that f is continuous by constructing an obviously continuous function $\tilde{f}: \mathbb{R} \to Y$ such that the restriction of $\tilde{f}$ to X coincides with f. Indeed, let $\tilde{f}: \mathbb{R} \to [0,2]$ be given by

$$\tilde{f}(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \le x \le 1, \\ 1 & \text{if } 1 < x \le 2, \\ x-1 & \text{if } 2 < x \le 3, \text{ and} \\ 2 & \text{if } 3 < x. \end{cases}$$

If one draws the graph of $\tilde{f}$ one sees that it is continuous; a rigorous proof is tedious rather than difficult, and so we omit it.



On the other hand, the inverse function g is not continuous. The intuitive reason for this is that g breaks the interval into two pieces. To prove it rigorously, note first that since X and Y are subspaces of $\mathbb{R}$, the rules for the subspace topology apply: a subset of X is open if and only if it has the form $X \cap U$ where U is an open subset of $\mathbb{R}$, and a subset of Y is open if and only if it has the form $Y \cap U$ with U open in $\mathbb{R}$. Thus $X \cap (1/2, 3/2) = (1/2, 1]$ is open in X. Now $g^{-1}((1/2, 1]) = (1/2, 1]$, and this is not an open subset of Y: the point $1 \in (1/2, 1]$ is not an interior point of $(1/2, 1]$ since every open ball $B(1, \varepsilon)$ (where $\varepsilon > 0$) contains points of $Y = [0, 2]$ that are not in $(1/2, 1]$. Thus it is not true that U open in X implies that $g^{-1}(U)$ is open in Y; so g is not continuous.