Subspace topology

Let $(X, \mathcal{U})$ be a topological space. That is, $\mathcal{U}$ is a collection of subsets of X satisfying
(T1) X, $\emptyset \in \mathcal{U}$,
(T2) whenever $(A_i)_{i \in I}$ is a family of sets in $\mathcal{U}$, then $\bigcup_{i \in I} A_i \in \mathcal{U}$, and
(T3) whenever A, $B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.
(Note that (T3) is equivalent to the condition that the intersection of any finite collection of elements of $\mathcal{U}$ is in $\mathcal{U}$, as can easily be proved by induction.)

Suppose now that S is any subset of X, and put $\mathcal{V} = \{ S \cap A \mid A \in \mathcal{U} \}$. It is not hard to prove that $\mathcal{V}$ is a topology on S. Firstly, since $S \subseteq X$ we have $S \cap X = S$, while it is trivial that $S \cap \emptyset = \emptyset$. Since X, $\emptyset \in \mathcal{U}$ (since $\mathcal{U}$ satisfies (T1)), it follows that S, $\emptyset \in \mathcal{V}$. So (T1) holds for $\mathcal{V}$. Next, suppose that $(B_i)_{i \in I}$ is a family of sets in $\mathcal{V}$. For each $i \in I$ there is an $A_i \in \mathcal{U}$ with $B_i = S \cap A_i$. Now $\bigcup_{i \in I} A_i \in \mathcal{U}$, by (T2) for $\mathcal{U}$, and since

$$S \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} S \cap A_i = \bigcup_{i \in I} B_i$$

it follows that $\bigcup_{i \in I} B_i \in \mathcal{V}$. Hence $\mathcal{V}$ satisfies (T3). Finally, if P, Q are arbitrary elements of $\mathcal{V}$ then $P = S \cap A$ and $Q = S \cap B$ for some A, $B \in \mathcal{U}$, and we see that

$$P \cap Q = (S \cap A) \cap (S \cap B) = S \cap (A \cap B) \in \mathcal{V}$$

since $A \cap B \in \mathcal{U}$. So $\mathcal{V}$ also satisfies (T3), and thus is a topology on S.

Definition. Let $(X, \mathcal{U})$ be a topological space and S a subset of X The topology $\mathcal{V}$ on the set S defined by $\mathcal{V} = \{ S \cap A \mid A \in \mathcal{U} \}$ (as above) is called the topology on S *induced* by the topology $\mathcal{U}$ on X. A *topological subspace* of $(X, \mathcal{U})$ is a topological space of the form $(S, \mathcal{V})$, where S is a subset of X and $\mathcal{V}$ the induced topology. The induced topology is also sometimes called the *relative topology*, or the *subspace topology*. A subset of X is said to be *open relative to* S if it is an open set of the subspace topology (so that it is of the form $S \cap A$ for some $A \in \mathcal{U}$).

Recall that if (X, d) is a metric space then there is a standard topology on X derived from the metric: it consists of those subsets U of X such that for all $a \in U$ there is an $\varepsilon > 0$ such that $B_d(a, \varepsilon) \subseteq U$. Furthermore, if S is any subset of X and d' the restriction of d to S, then (S, d') is a metric space. (We call d' the metric induced by d.) Now we can obtain a topology on S in either of two ways: the topology on X derived from the metric d induces a topology $\mathcal{V}_\infty$ on S, and there is a topology $\mathcal{V}_\in$ on S derived from the induced metric d'. One would hope that $\mathcal{V}_1 = \mathcal{V}_2$, and this is indeed true. On the one hand, suppose that $A \in \mathcal{V}_2$. This means that for all $a \in A$ there is a positive number μ_a such that $B_{d'}(a, \mu_a) \subseteq A$. Now

$$\begin{aligned} B_{d'}(a, \mu_a) = \{ x \in S \mid d'(x, a) < \mu_a \} &= \{ x \in S \mid d(x, a) < \mu_a \} \\ &= S \cap \{ x \in X \mid d(x, a) < \mu_a \} = S \cap B_d(a, \mu_a); \end{aligned}$$

Moreover, $A \subseteq \bigcup_{a \in A} B_{d'}(a, \mu_a)$ (since $a \in B_{d'}(a, \mu_a)$ for each a), and $\bigcup_{a \in A} B_{d'}(a, \mu_a) \subseteq A$ (since $B_{d'}(a, \mu_a) \subseteq A$ for each a, by the choice of μ_a). Thus

$$A = \bigcup_{a \in A} B_{d'}(a, \mu_a) = \bigcup_{a \in A} S \cap B_d(a, \mu_a) = S \cap \bigcup_{a \in A} B_d(a, \mu_a),$$

which is an open set of the induced topology $\mathcal{V}_1$, since $\bigcup_{a\in A} B_d(a,\mu_a)$ is an open subset of X. On the other hand, suppose that $A \in \mathcal{V}_1$, so that $A = S \cap U$ for some open subset U of X. Since U is open, for each $a \in U$ there is a $\mu > 0$ such that $B_d(a,\mu) \subseteq U$; in particular, such a *mu* exists for each $a \in A$ (since $A \subseteq U$), and we find that

$$B_{d'}(a,\mu) = S \cap B_d(a,\mu) \subseteq S \cap U = A,$$

which shows that $A \in \mathcal{V}_2$.

The following result (for metric spaces) appears as Theorem 3.1 on p. 52 of Choo's notes. Note, however, that there is a misprint: the important assumption that f is continuous was accidentally omitted. We prove the result here in the more general context of topological spaces.

Proposition. *Let X, Y be topological spaces and $f\colon X \to Y$ a continuous mapping. Let S be any subspace of X, and $f_S\colon S \to Y$ the restriction of f. Then f_S is continuous.*

Proof. Let U be an open subset of Y. By definition,

$$\begin{aligned} f_S^{-1}(U) = \{\, x \in S \mid f_S(x) \in U \,\} &= \{\, x \in S \mid f(x) \in U \,\} \\ &= S \cap \{\, x \in X \mid f(x) \in U \,\} = S \cap f^{-1}(U). \end{aligned}$$

Now $f^{-1}(U)$ is an open subset of X since U is open in Y and $f\colon X \to Y$ is continuous. So $S \cap f^{-1}(U)$ is an open subset of S (in the subspace topology). Thus we have shown that $f_S^{-1}(U)$ is open in S whenever U is open in Y; hence f_S is continuous. □

A similarly straightforward result says that the composite of two continous functions is always continuous.

Proposition. *If X, Y and Z are topological spaces, and $f\colon X \to Y$ and $g\colon Y \to Z$ continuous functions, then the function $g \circ f\colon X \to Z$ (defined by $(g\circ f)(x) = g(f(x))$ for all $x \in X$) is continuous.*

Proof. Our task is to show that $(g \circ f)^{-1}(U)$ is open in X whenever U is open in Z.

Let $U \subseteq Z$ be open. Then

$$\begin{aligned} (g\circ f)^{-1}(U) &= \{\, x \in X \mid (g\circ f)(x) \in U \,\} \\ &= \{\, x \in X \mid g(f(x)) \in U \,\} \\ &= \{\, x \in X \mid f(x) \in g^{-1}(U) \,\} \\ &= f^{-1}(g^{-1}(U)). \end{aligned}$$

Since g is continuous and U is open it follows that $g^{-1}(U)$ is open. Now since f is continuous it follows that $f^{-1}(g^{-1}(U))$ is open. So we have shown that $(g\circ f)^{-1}(U)$ is open whenever U is open, as required. □

Bases

If X and Y are topological spaces then there is a natural way to make the Cartesian product $X \times Y = \{\, (x,y) \mid x \in X \text{ and } y \in Y \,\}$ into a topological space. Before we can discuss this we need to introduce another concept.

Definition. Let X be a topological space. A collection $\mathcal{B}$ of open subsets of X is called a *base* for the topology on X if every open set can be expressed as a union of sets in $\mathcal{B}$.

Example. In $\mathbb{R}$ the open intervals form a base for the topology. More generally, in any metric space the open balls form a base for the topology. To prove this one must show that every open set is expressible as a union of the open balls. The proof of this was incorporated in one of our proofs above, but it will do us no harm to repeat it!

Proposition. *If X is a metric space and $U \subseteq X$ is open, then U is the union of the open balls it contains.*

Proof. On the one hand, the union of all the open balls contained in U is obviously a subset of U; on the other, if $x \in U$ is arbitrary then $x \in \mathrm{Int}(U)$ (as every point of an open set is an interior point), hence x lies in an open ball contained in U, and hence x is in the union of all the open balls contained in U. □

In many cases when it is desirable to make a set into a topological space, the most convenient way to do so is to specify a base for the topology, rather than attempt to describe all open sets directly. The situation with metric spaces illustrates this: open sets are defined in terms of open balls. One could perhaps manage to give a reasonable discussion of metric spaces without using the concept of an open set, but one could not sensibly avoid talking about open balls.

Note that a base for a topology determines the topology uniquely: there cannot be two different topologies on one set X sharing a common base $\mathcal{B}$. This is because the open sets of the topology can be characterized as those sets that are unions of sets in $\mathcal{B}$. (The definition of the concept of a base says that all open sets are unions of sets in $\mathcal{B}$; on the other hand, since the elements of $\mathcal{B}$ are themselves open sets and the union of any collection of open sets is open, it is also true that every set which is a union of sets in $\mathcal{B}$ is an open set.) However, it is not the case that every collection of subsets of an arbitrary set X can serve as a base for a topology on X. This is because the intersection of two open sets has to be open, and it is clear that if $\mathcal{B}$ is an arbitrary collection of subsets of X then there is no guarantee that the intersection of any two elements of $\mathcal{B}$ will be expressible as a union of elements of $\mathcal{B}$. Provided that the collection $\mathcal{B}$ does have this property, and provided that the elements of $\mathcal{B}$ cover X, then it will be the case that the collection $\mathcal{B}$ determines a topology on X.