Uniform and pointwise convergence

Definition. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on a set S, and f also a real-valued function on S. We say that $f_n \to f$ *uniformly* on S if for all $\varepsilon > 0$ there exists an $N \in \mathbb{Z}$ such that $|f_n(s) - f(s)| < \varepsilon$ for all $s \in S$ for all $n > N$.

We say that $f_n \to f$ *pointwise* on S if for all $s \in S$ and all $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that $|f_n(s) - f(s)| < \varepsilon$ for all $n > N$.

The difference between the two concepts is that for pointwise convergence the number N is allowed to depend on the element $s \in S$, whereas for uniform convergence N depends only on ε, and the one N must work for all $s \in S$. Uniform convergence is thus a stronger condition than pointwise convergence: uniform convergence implies pointwise convergence, but not vice versa (unless the set S is finite).

Let $\mathcal{B} = \mathcal{B}[a, b]$ be the set of all bounded real-valued functions on the interval $[a, b]$ in $\mathbb{R}$, and for f, $g \in \mathcal{B}$ define

$$d(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

Recall that this is a metric, known as the sup metric, or uniform metric, on $\mathcal{B}$. It is called the uniform metric because convergence in the metric space $(\mathcal{B}, d)$ is uniform convergence on $[a, b]$: a sequence $(f_k)_{k=1}^{\infty}$ converges in $(\mathcal{B}, d)$ to an element $f \in \mathcal{B}$ if and only if $f_k \to f$ uniformly on $[a, b]$ as $k \to \infty$. To see this, first observe that the condition

$$d(f_k, f) \le \mu \tag{1}$$

is equivalent to the condition

$$|f_k(x) - f(x)| \le \mu \qquad \text{for all } x \in [a, b], \tag{2}$$

since $d(f_k, f)$ is the supremum of the set $\{\, |f_k(x) - f(x)| \;\big|\; x \in [a, b] \,\}$, and to say that the supremum of a set of numbers is less than or equal to μ is equivalent to saying that all the numbers in the set are less than or equal to μ. By definition, $f_k \to f$ in the space $(\mathcal{B}, d)$ if and only if for all $\varepsilon > 0$ there is an N such that

$$d(f_k, f) < \varepsilon \tag{1$'$}$$

for all $k > N$. So it is fairly clear that $f_k \to f$ in the space $(\mathcal{B}, d)$ if and only if for all $\mu > 0$ there is an N such that condition (1) above holds for all $k > N$; the point is that if $(1)'$ holds for some ε then (1) holds with $\mu = \varepsilon$, while if $(1')$ holds for some μ then (1) holds with $\varepsilon = 2\mu$. Since (1) and (2) are equivalent, this tells us that $f_k \to f$ in the space $(\mathcal{B}, d)$ if and only if for all $\mu > 0$ there is an N such that (2) holds for all $n > N$. But again it is clear that this is equivalent to the statement that for all $\varepsilon > 0$ there is an N such that

$$|f_k(x) - f(x)| < \varepsilon \qquad \text{for all } x \in [a, b] \tag{2$'$}$$

for all $k > N$: if $(2)'$ holds then (2) holds with $\mu = \varepsilon$, and if (2) holds then $(2)'$ holds with $\varepsilon = 2\mu$. But the condition that for all $\varepsilon > 0$ there is an N such that (2) holds for all $n > N$ is exactly the definition of the statement that $f_k \to f$ uniformly on $[a, b]$; so we have shown, as required, that the statement that $f_k \to f$ in $(\mathcal{B}, d)$ is equivalent to the statement that $f_k \to f$ uniformly.

A digression relating to Assignment 1

Let $(S,\mathcal{U})$ and $(T,\mathcal{V}$ be topological spaces. Recall that this means that $\mathcal{U}$ and $\mathcal{T}$ are collections of subsets of S and T respectively that are closed under arbitrary unions and finite intersections, and $\emptyset$, $S \in \mathcal{U}$ and $\emptyset$, $T \in \mathcal{V}$. The sets in the collection $\mathcal{U}$ are called the open subsets of S, and the sets in $\mathcal{V}$ the open subsets of T. In Lecture 2 we defined continuity for functions between metric spaces, and gave a characterization of continuity in terms of open sets. We use this characterization as the definition of continuity in the more general context of topological sapaces.

Definition. Let $(S,\mathcal{U})$ and $(T,\mathcal{V}$ be topological spaces, and $f: S \to T$ a function. We say that f is *continuous* at a point $a \in S$ if $a \in \mathrm{Int}(f^{-1}(V))$ for every open neighbourhood V of $f(a)$. Furthermore, we say that f is *continuous on* S if $f^{-1}(V)$ is an open subset of S whenever V is an open subset of T.

It is easy to show that f is continuous on S if and only if it is continuous at every $a \in S$; the argument needed was given in Lecture 2 in the context of metric spaces, and it applies unchanged for topological spaces in general.

It is obvious that if $\mathcal{U}$ is the set of all subsets of S then $\mathcal{U}$ is a topology on S. It is also true that if $\mathcal{U} = \{\emptyset, S\}$ then $\mathcal{U}$ is a topology on S. These are known, respectively, as the largest (or finest) and smallest (or coarsest) topologies on S. These are rather uninteresting toplogies, in fact, but nevertheless we need to know what our definition of continuity means in these extreme cases. Hence Question 2 of the assignment asks you to determine which functions from S to T are continuous in the following four cases: when $\mathcal{U}$ and $\mathcal{V}$ are the finest topologies on S and T respectively; when $\mathcal{U}$ is the finest on S and $\mathcal{T}$ the coarsest on T; when $\mathcal{U}$ is the coarsest and $\mathcal{V}$ the finest; when $calS$ and $calV$ are the coarsest toplogies on S and T.

More on convergence

There are some fairly straightforward general facts about convergence in metric spaces that we ought to prove.

Proposition. *Let (X, d) be a metric space, (x_n) and (y_n) sequences in X, and x, y points of X. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.*

Proof. By the triangle inequality,

$$d(x_n, y_n) \le d(x_n, x) + d(x, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n)$$
$$d(x, y) \le d(x, x_n) + d(x_n, y) \le d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

It follows that

$$d(x, y) - d(x, x_n) - d(y_n, y) \le d(x_n, y_n) \le d(x, y) + d(x_n, x) + d(y, y_n). \tag{3}$$

Since $d(x, x_n) = d(x_n, x) \to 0$ and $d(y, y_n) = d(y_n, y) \to 0$ as $n \to \infty$, the inequalities in (3) show that $d(x_n, y_n)$ is sandwiched between two quantities that both approach $d(x, y)$ as $n \to \infty$, and this gives the desired result. □

Corollary. *If a sequence of points in a metric space converges then its limit is unique.*

Proof. Suppose that (x_n) is a sequence in the metric space (X,d), and suppose that $x_n \to x$ and also $x_n \to y$ as $n \to \infty$, where x and y are distinct points of X. The proposition shows that $d(x_n,x_n) \to d(x,y)$ as $n \to \infty$; so, since $d(x,y)/2 > 0$, there exists $N \in \mathbb{Z}$ such that $|d(x,y) - d(x_n,x_n)| < d(x,y)/2$ for all $n > N$. But of course $d(x_n,x_n) = 0$; so we have shown that $d(x,y) < d(x,y)/2$, contradicting $d(x,y) > 0$. So it is not possible for (x_n) to have two distinct limits x and y. □

We also also have the following characterization of the closure of a set in terms of limits of sequences of points in the set.

Proposition. *Let (X,d) be a metric space, $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if there exists a sequence $(a_n)_{n=1}^{\infty}$ of points of A with $a_n \to x$ as $n \to \infty$.*

Proof. Let $x \in \overline{A}$. For each $\varepsilon > 0$ the intersection $B_d(x,\varepsilon) \cap A \neq \emptyset$ (by the first lemma in Lecture 5). So for each $n \in \mathbb{Z}^+$ we can choose an element $a_n \in B_d(x,1/n) \cap A$, and then

$$0 \le d(x,a_n) < 1/n \to 0 \qquad \text{as } n \to \infty,$$

showing that $a_n \to x$ as $n \to \infty$. So elements of $\overline{A}$ are limits of sequences in A.

Conversely, suppose that $a_n \in A$ for all $n \in \mathbb{Z}^+$, and $a_n \to x$ as $n \to \infty$. Let U be an arbitrary open neighbourhood of x. Then there exists $\varepsilon > 0$ such that $B_d(x,\varepsilon) \subseteq U$, and since $\lim_{n\to\infty} d(a_n,x) = 0$ there exists $N \in \mathbb{Z}^+$ such that $d(a_n,x) < \varepsilon$ for all $n \ge N$. In particular, $a_N \in B_d(x,\varepsilon) \subseteq U$, showing that $U \cap A \neq \emptyset$. But U was arbitrary; so we have shown that all open neighbourhoods of x have nonempty intersection with A. By the lemma from Lecture 5 it follows that $x \in \overline{A}$; so limits of convergent sequences in A are always elements of $\overline{A}$. □

Convergence in topological spaces

So we can naturally extend the concept of convergence to an arbitrary topological space as follows: if $(X,\mathcal{U})$ is a topological space then a sequence $(x_n)_{n=1}^{\infty}$ in X is said to converge to $x \in X$ if for every open neighbourhood U of x there exists an $N \in \mathbb{Z}$ such that $x_n \in U$ for all $n > N$.

However, in this very general setting convergence is not as well-behaved as it is for metric spaces. For example, it is possible for one sequence to converge to two different points. The problem with arbitrary topological spaces, in this regard, is that there may not be enough open sets to distinguish points. It is quite possible to have two distinct points x and y such that every open set containing x also contains y. If this happens then the constant sequence (x_n) defined by $x_n = y$ for all n converges to x as well as to y. One could reasonably take the view that the axioms of topological spaces ought to be strengthened in order to prohibit this phenomenon, since it does seem very desirable that limits should be unique. In fact, there are many different alternative "separation properties" that have been studied; they are all a little different from each other: there are topological spaces which satisfy some of them and not others. We shall investigate this a little next time.