Definition. Let S be a set and $\mathcal{U}$ a collection of subsets of S. We call $\mathcal{U}$ a *topology* for S if the following properties hold.

a) $\emptyset \in \mathcal{U}$ and $S \in \mathcal{U}$.
b) If I is any indexing set and $U_i \in \mathcal{U}$ for all $i \in I$ then $\bigcup_{i\in I} U_i \in \mathcal{U}$.
c) If I is any finite indexing set and $U_i \in \mathcal{U}$ for all $i \in I$ then $\bigcap_{i\in I} U_i \in \mathcal{U}$.

A *topological space* is a pair $(S,\mathcal{U})$ consisting of a set S and a topology $\mathcal{U}$ for S. The elements of $\mathcal{U}$ are called the *open sets* of the topology.

By far the most important examples of topological spaces are metric spaces with the topology of open sets defined in terms of open balls, as we have described. However, there are very important topologies which are not of this kind. One such is the *Zariski topology* on $\mathbb{C}^2$, which we now proceed to describe.†

A polynomial in two variables x and y is simply an expression built up using addition and multiplication, and scalars, x's and y's, For example, $\sqrt{2} + 5x + \pi x^3 y^2$ is a polynomial. Suppose that $p_i(x, y)$, $i \in I$, is any indexed family of polynomials in x and y with coefficients in $\mathbb{C}$. The zero set of this family is the set

$$\{\, (a, b) \in \mathbb{C}^2 \mid p_i(a, b) = 0 \text{ for all } i \in I \,\}.$$

Let us say that a subset of $\mathbb{C}^2$ is *closed* if it is the zero set of some family of complex polynomials in x and y, and define a subset of $\mathbb{C}^2$ to be open if its complement is closed. The open sets thus defined constitute the Zariski topology on $\mathbb{C}^2$. (In fact, making some fairly natural changes, instead of $\mathbb{C}^2$ we could have used F^n for any field F and any positive integer n.) The Zariski topology plays a key role in the subject known as *algebraic geometry*, which is beyond the scope of this course.

We proceed now to define some standard concepts used in the study of topological spaces. The student is firmly encouraged to always think of metric spaces, and in particular $\mathbb{R}^2$ with the usual metric, when trying to understand these concepts.

Definition. Let $(X,\mathcal{U})$ be a topological space. A subset A of X is said to be *closed* (relative to this topology) if its complement $X \setminus A$ is open.

Usually, most subsets of X are neither open nor closed. It is also possible for a set to be both open and closed: for example, X and $\emptyset$ are both open and closed. But for most common spaces X and $\emptyset$ are the only sets that are both open and closed.

Since the open sets of a topology are just the complements of the closed sets, a topology can equally well be specified by giving the closed sets rather then the open sets. (Indeed, for the Zariski topology it is easier to describe the closed sets than the open sets.) In terms of closed sets the defining properties of a topology on X can be stated as follows:

a′) $\emptyset$ and X are both closed;
b′) the intersection of any collection of closed sets is closed;
c′) the union of any finite collection of closed sets is closed.

Definition. Let $(X,\mathcal{U})$ be a topological space. An *interior point* of a set $A \subseteq X$ is a point $a \in A$ such that there is an open set U with $U \subseteq A$ and $a \in U$. The set of all

† This example was not done in the lecture, and is included here only for interest. It is not part of the syllabus of the course.

interior points of A is called the *interior* of A, denoted by $\mathrm{Int}(A)$. A *neighbourhood* of a point $a \in X$ is a set $A \subseteq X$ such that $a \in \mathrm{Int}(A)$. An *open neighbourhood* of a point a is an open set containing a.

Let $(X,\mathcal{U})$ be a topological space, and $A \subseteq X$. Let Q be the union of all the open sets U which are contained in A. That is, $Q = \bigcup\{\, U \in \mathcal{U} \mid U \subseteq A \,\}$. As a union of open sets, the set Q must be open (by condition (b) in the definition of a topology). As a union of subsets of A, the set Q must be a subset of A. Thus Q has the properties that

- Q is open and $Q \subseteq A$, and
- if U is open and $U \subseteq A$ then $U \subseteq Q$.

So we can say that Q is the largest open set which is a subset of A.

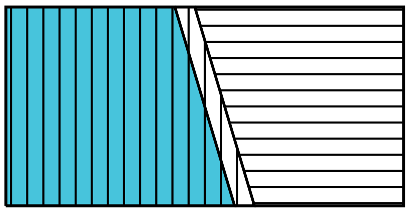
It is fairly clear that in fact $Q = \mathrm{Int}(A)$. Firstly, all points of Q are interior points of A, since Q is open and $Q \subseteq A$. On the other hand, if a is an interior point of A then $a \in U$ for some open U with $U \subseteq A$, and since $U \subseteq Q$ this gives $a \in Q$. So all interior points of A are in Q.

So $\mathrm{Int}(A)$, the interior of A, is the union of all open sets contained in A. Somewhat analogously, we define the *closure* $\overline{A}$ of A to be the intersection of all closed sets containing A. We see that

- $\overline{A}$ is closed, and $A \subseteq \overline{A}$, and
- if C is any closed set with $A \subseteq C$ then $\overline{A} \subseteq C$.

Thus $\overline{A}$ is the smallest closed set containing A.

Let A be an arbitrary subset of X and put $B = X \backslash A$, so that A and B are complements of each other. If S is any subset of A then B is a subset of the complement of S. If S is open then its complement is closed, and vice versa. The following diagram illustrates the situation:



The rectangle represents the metric space X, the region hatched with vertical lines the subset A and the region hatched with horizontal lines the subset B (the complement of A). The blue part represents a subset S of A that is assumed to be an open set of X; its complement (white) is then clearly a closed set containing B.

Clearly from this we can say that the complement of the largest open set contained in A is the smallest closed set containing B. That is, if B is the complement of A then the closure of B is the complement of the interior of A. (And, symmetrically, the interior of B is the complement of the closure of A.)