The definition of continuity (as stated in Lecture 1 for functions from $\mathbb{R}^2$ to $\mathbb{R}^2$) makes sense for functions from any metric space (X, d) to any other metric space (Y, d'):

> A function $f: X \to Y$ is continuous at the point $a \in X$ if for every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that the following condition holds: for all $x \in X$, if $d(x, a) < \delta$ then $d'(f(x), f(a)) < \varepsilon$.

Using the concept of "open ball", this can be rephrased as follows:

> A function $f: X \to Y$ is continuous at $a \in X$ if and only if for every open ball B with centre at $f(a)$ there is an open ball C with centre a such that $f(C) \subseteq B$.

Note that the condition $f(C) \subseteq B$ is equivalent to $C \subseteq f^{-1}(B)$. (This is easy to prove: it follows immediately from the definitions of "image" and "preimage".)

The following proposition generalizes the above statement slightly.

Proposition. *Let (X, d), (Y, d') be metric spaces and $f: X \to Y$ a function, and let $a \in X$. Then f is continuous at a if and only if for every open subset U of Y with $a \in f^{-1}(U)$ there is an open ball C with centre a such that $C \subseteq f^{-1}(U)$.*

Proof. Suppose first that f satisfies the stated condition; we shall show that f is continuous at a.

Let $\varepsilon > 0$. Then $U = B(f(a), \varepsilon)$ is an open subset of Y, and $a \in f^{-1}(U)$ (since $f(a) \in U$). So by the given condition there exists an open ball C centred at a such that $C \subseteq f^{-1}(U)$. Let δ be the radius of C (so that $C = B(a, \delta)$). Now if x is an arbitrary element of X satisfying $d(x, a) < \delta$, then

$$x \in C \subseteq f^{-1}(U),$$

whence $f(x) \in U = B(f(a), \varepsilon)$, which means that $d'(f(x), f(a)) < \varepsilon$.

Thus we have have shown that for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $x \in X$, if $d(x, a) < \delta$ then $d(f(x), f(a)) < \varepsilon$. That is, we have shown that f is continuous at a.

Conversely, suppose that f is continuous at a, and let U be an open subset of Y such that $a \in f^{-1}(U)$. Since U is open and $f(a) \in U$ there is an $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subseteq U$. Since f is continuous at a there exists $\delta > 0$ such that, for all $x \in X$, if $d(x, a) < \delta$ then $d'(f(x), f(a)) < \varepsilon$. Now put $C = B(a, \delta)$, an open ball centred at a. For all $x \in C$ we have $d(x, a) < \delta$, which gives $d'(f(x), f(a)) < \varepsilon$, and hence $f(x) \in B(f(a), \varepsilon) \subseteq U$. So $x \in f^{-1}(U)$ whenever $x \in C$; in other words, $C \subseteq f^{-1}(U)$. Thus we have shown that for every open set U containing $f(a)$ there is an open ball centred at a and contained in $f^{-1}(U)$, as required. □

In view of the definition of the interior of a set, we can restate the above result as follows.

Corollary. *The function $f: X \to Y$ is continuous at a if and only if, for all open subsets U of Y, if $a \in f^{-1}(U)$ then $a \in \operatorname{Int}(f^{-1}(U))$.*

This enables us to now give a concise characterization of continuous functions.

Corollary. *If (X, d) and (Y, d') are metric spaces then a function $f: X \to Y$ is continuous if and only if $f^{-1}(U)$ is an open subset of X whenever U is an open subset of Y.*

Proof. To say that f is continuous is to say that it is continuous at all points $a \in X$. By the previous corollary, this holds if and only if for all open $U \subset Y$ and all $a \in X$, if

$a \in f^{-1}(U)$ then $a \in \mathrm{Int}(f^{-1}(U))$. That is, for every open $U \subseteq Y$, all points of $f^{-1}(U)$ are interior points. But to say that all points of $f^{-1}(U)$ are interior points is to say that $f^{-1}(U)$ is open. □

Some inequalities

Suppose that $0 \le \theta \le 1$. If (x_0, y_0) and (x_1, y_1) are points in $\mathbb{R}^2$ then the point (x, y) defined by

$$\begin{aligned} x &= \theta x_0 + (1-\theta)x_1 \\ y &= \theta y_0 + (1-\theta)y_1 \end{aligned}$$

lies on the line segment joining (x_0, y_0) and (x_1, y_1). Now the graph of $y = \ln x$ is concave downwards; so if (x_0, y_0) and (x_1, y_1) are on this graph then (x, y) will be below it; that is, $y \le \ln x$. In other words, if a, $b > 0$ and we define

$$\begin{aligned} x_0 &= a \\ y_0 &= \ln a \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 &= b \\ y_1 &= \ln b \end{aligned}$$

so that

$$\begin{aligned} x &= \theta a + (1-\theta)b \\ y &= \theta(\ln a) + (1-\theta)(\ln b) \end{aligned}$$

then it follows that

$$\theta(\ln a) + (1-\theta)(\ln b) \le \ln(\theta a + (1-\theta)b).$$

Taking exponentials of both sides, using the fact that e^x is monotone increasing, it follows that

$$e^{\theta(\ln a)+(1-\theta)(\ln b)} \le \theta a + (1-\theta)b.$$

But $e^{\theta(\ln a)+(1-\theta)(\ln b)} = e^{\theta(\ln a)}e^{(1-\theta)(\ln b)} = a^\theta b^{1-\theta}$; so we have shown that

$$a^\theta b^{1-\theta} \le \theta a + (1-\theta)b. \tag{$*$}$$

for all a, $b > 0$. The same in fact holds for a, $b \ge 0$, since if either a or b is zero then the left hand side is zero, while the right hand side remains nonnegative.

Hölder's Inequality. *Let $p > 1$ and put $q = p/(p-1)$ (so that $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$). Let a_k, b_k be arbitrary complex numbers, where k runs from 1 to n. Then*

$$\sum_{k=1}^{n} |a_k b_k| \le \Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/p} \Big(\sum_{k=1}^{n} |b_k|^q\Big)^{1/q}.$$

Proof. Let $c_k = |a_k|^p$ and $d_k = |b_k|^q$, and put $C = \sum_{k=1}^n c_k$ and $D = \sum_{k=1}^n d_k$. Put $\theta = 1/p$, so that $1 - \theta = 1/q$, and apply $(*)$ with c_k/C in place of a and d_k/D in place of b. We obtain

$$(c_k/C)^{1/p}(d_k/D)^{1/q} \le (1/p)(c_k/C) + (1/q)(d_k/D).$$

Summing from $k = 1$ to n gives

$$\sum_{k=1}^{n} \frac{c_k^{1/p} d_k^{1/q}}{C^{1/p} D^{1/q}} \le \frac{1}{pC} \sum_{k=1}^{n} c_k + \frac{1}{qD} \sum_{k=1}^{n} d_k$$
$$= \frac{1}{p} + \frac{1}{q} = 1.$$

Hence $\sum_{k=1}^{n} c_k^{1/p} d_k^{1/q} \le C^{1/p} D^{1/q}$; that is,

$$\sum_{k=1}^{n} |a_k b_k| \le \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q \right)^{1/q}$$

as required. □

The special case of Hölder's Inequality in which $p = q = 2$ is known as *Cauchy's Inequality.*

Minkowski's Inequality. *Let $p \ge 1$, and let a_k, $b_k \in C$ be arbitrary. Then*

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p \right)^{1/p} \le \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{1/p}.$$

Proof. Since $|a+b| \le |a| + |b|$ for all complex numbers a and b, it is clear that the result holds for $p = 1$. So we assume that $p > 1$. Put $q = p/(p-1)$.

For all k from 1 to n we have

$$(a_k + b_k)^p = a_k (a_k + b_k)^{p-1} + b_k (a_k + b_k)^{p-1}$$

and so using standard properties of the modulus function for complex numbers (namely $|ab| = |a||b|$ and $|a + b| \le |a| + |b|$ for all a, $b \in \mathbb{C}$, and $|a^t| = |a|^t$ for all $a \in \mathbb{C}$ and $t \in \mathbb{R}$) we deduce that

$$|a_k + b_k|^p \le |a_k|(|a_k + b_k|)^{p-1} + |b_k|(|a_k + b_k|)^{p-1}$$

for all k. Summing from $k = 1$ to n, and then applying Hölder's Inequality to each of the sums on the right hand side gives

$$\sum_{k=1}^{n} |a_k + b_k|^p \le \sum_{k=1}^{n} |a_k|(|a_k + b_k|)^{p-1} + \sum_{k=1}^{n} |b_k|(|a_k + b_k|)^{p-1}$$
$$\le \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} (|a_k + b_k|)^{(p-1)q} \right)^{1/q} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} (|a_k + b_k|)^{(p-1)q} \right)^{1/q}$$
$$= \left(\left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{1/p} \right) \left(\sum_{k=1}^{n} (|a_k + b_k|)^p \right)^{1/q},$$

where in the last line we have used $(p-1)q = p$. Dividing through by the second factor on the right hand side gives

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p \right)^{1-(1/q)} \le \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{1/p},$$

which is the required result, since $1 - (1/q) = 1/p$. □