THE UNIVERSITY OF SYDNEY
PURE MATHEMATICS 3901

Metric Spaces 2000

Tutorial 10

1. Determine which of the following subsets A are compact subsets in the appropriate $\mathbb{R}^n$.

(*i*) $A = [0, 2)$

(*ii*) $A = \mathbb{Q} \cap [0, 1]$

(*iii*) $A = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0 \}$

(*iv*) $A = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 4 \} \cup \{(1, 2)\}$.

Solution.

(*i*) Compact sets are closed (in any Hausdorff space). Since $[0, 2)$ is not closed it is not compact. (The open covering $[0, 2) \subseteq \bigcup_{n=1}^{\infty}(-1, 2 - \frac{1}{n})$ has no finite subcovering.)

(*ii*) This set is not closed, and so it is not compact. (And, for example $A \subseteq \bigcup_{n=1}^{\infty}((-1, \frac{(n-1)\sqrt{2}}{2n}) \cup (\frac{(n+1)\sqrt{2}}{2n}, 2))$ is an open covering for which there is no finite subcovering.)

(*iii*) To be compact a set must be bounded; this set is not.

(*iv*) This set is closed and bounded; so it is compact (by Heine-Borel).

2. Let A and B be compact subsets of a topological space X. Show that $A \cap B$ and $A \cup B$ are also compact.

Solution.

The question is wrong as stated: to prove that $A \cap B$ is compact an extra assumption is needed. However, $A \cup B$ is necessarily compact. For suppose that $(V_i)_{i \in I}$ is a family of open sets with $A \cup B \subseteq \bigcup_{i \in I} V_i$. Since $A \subseteq A \cup B$ it follows that $A \subseteq \bigcup_{i \in I} V_i$, and since A is compact there is a finite subset J of I with $A \subseteq \bigcup_{i \in J} V_i$. Similarly there is a finite subset K of I with $B \subseteq \bigcup_{i \in J} V_i$. So $A \cup B \subseteq \bigcup_{i \in J} V_i \cup \bigcup_{i \in K} V_i = \bigcup_{i \in J \cup K} V_i$. Since J and K are finite, so is $J \cup K$, and we have produced a finite subcovering from an arbitrary open covering of $A \cup B$. So $A \cup B$ is compact.

If the space X is assumed to be Hausdorff then the set A is closed, since compact implies closed in Hausdorff spaces (as we proved in lectures). It then follows from Question 4 below that $A \cap B$ is compact. (Note that the result in Question 4 is valid for all topological spaces, not just Hausdorff ones.)

3. Let $(A_i)_{i \in I}$ be any family of compact subsets of a metric space (X, d). Prove that $B = \bigcap_{i \in I} A_i$ is compact, while $\bigcup_i A_i$ is not necessarily compact.

Solution.

An infinite union obviously need not be bounded. For example, in $\mathbb{R}$ let $A_i = [-i, i]$, for each positive integer i. Each A_i is compact, by Heine-Borel, but $\bigcup_{i=1}^{\infty} A_i = \mathbb{R}$ is not bounded, so not compact.

For the other part it suffices to assume that X is a Hausdorff space. (This is weaker than assuming that X is a metric space: all metric spaces are Hausdorff, but there are Hausdorff spaces that are not metrizable.) Again, we can use Question 4: choose a fixed $i_0 \in I$, put $A = \bigcap_{i \neq i_0} A_i$ and put $B = A_{i_0}$. Since X is Hausdorff the sets A_i are all closed, and so A is closed (being an intersection of closed sets). Since B is compact it follows that $A \cap B$ is compact. But $A \cap B = \bigcap_{i \in I} A_i$.

4. Let A and B be subsets of a topological space X such that A is closed and B is compact. Show that $A \cap B$ is compact.

Solution.

Let $(V_i)_{i \in I}$ be a family of open sets such that $A \cap B \subseteq \bigcup_{i \in I} V_i$. Let J be a set obtained by adding one more element to I: say $J = I \cup \{j\}$. Define $V_j = X \setminus A$, and observe that V_j is open since A is closed. Now for all $b \in B$, we have either that $b \in V_j$ (if $b \notin A$), or else $b \in A \cap B \subseteq \bigcup_{i \in I} V_i$, giving $b \in V_i$ for some $i \in I$. In either case $b \in V_i$ for some $i \in J$. So the family of sets $(V_i)i \in J$ form an open covering of B, and since B is compact there exists a finite subset L of J with $B \subseteq \bigcup_{i \in L} V_i$. Now the set $L \setminus \{j\}$ is a finite subset of $J \setminus \{j\} = I$, and we can show that $A \cap B \subseteq \bigcup_{i \in L \setminus \{j\}} V_i$. For suppose that $b \in A \cap B$. Then $b \in B \subseteq \bigcup_{i \in L} V_i$, and so $b \in V_i$ for some $i \in L$. But $b \in A$, and so $b \notin X \setminus A = V_j$. So $b \in V_i$ for some $i \in L \setminus \{j\}$, as required. Thus the arbitrarily chosen open covering $(V_i)_{i \in I}$ of the set $A \cap B$ has a finite subcovering, namely $(V_i)_{i \in L \setminus \{j\}}$. Hence $A \cap B$ is compact.

5. Let X be a non-empty set with d the standard discrete metric, and A any subset of X. Show that A is compact if and only if A is finite.

Solution.

Recall that this metric satisfies $d(x, y) = 1$ whenever $x \neq y$. It follows that for every $x \in X$ the open ball $B(x, \frac{1}{2})$ is just the singleton set $\{x\}$. Now every subset A of X can be expressed as a union of open balls; specifically, $A = \bigcup_{x \in A}\{x\}$. So all subsets of X are open. This condition implies that the compact sets are precisely the finite sets.

Firstly, suppose that A is a finite set, and suppose that $A \subseteq \bigcup_{i \in I} V_i$, where the V_i are any subsets of X. For each $a \in A$ we have $a \in \bigcup_{i \in I} V_i$, and so we may choose an element $i_a \in I$ such that $a \in V_{i_a}$. Then $A \subseteq \bigcup_{a \in A} V_{i_a}$, and since A is a finite set there are only finitely many terms in this union. So we have shown that every covering of A has a finite subcovering, and so A is compact.

Conversely, suppose that A is a compact subset of X. Then the singleton sets $(\{x\})_{x\in X}$ form an open covering of A, since $A \subseteq X = \bigcup_{x\in X}\{x\}$, and since the singleton sets are open. By the compactness of A there is a finite subcovering; that is, there is a finite subset B of X such that $A \subseteq \bigcup_{x\in B}\{x\} = B$. Since A is a subset of the finite set B, it too is finite.

6. Let X be a compact metric space (or topological space), and A any infinite subset of X. Show that A has an accumulation point in X. (That is, show that $A' \neq \emptyset$).

Solution.

Let A be a subset of X with no accumulation points in X. We shall show that A is finite.

Let $x \in X$ be arbitrary. Since x is not an accumulation point of A there is an open neighbourhood U_x of x such that $U_x \cap A \setminus \{x\} = \emptyset$. Equivalently, $U_x \cap A \subseteq \{x\}$. Since $x \in U_x$ it follows that $X \subseteq \bigcup_{x\in X} U_x$. That is, the family $\mathcal{C} = (U_x)_{x\in X}$ is an open covering of X. Since X is compact, $\mathcal{C}$ has a finite subcovering: there exists a finite subset $\{x_1, x_2, \dots, x_m\}$ of X with $X \subseteq \bigcup_{i=1}^m U_{x_i}$. (Indeed, this union equals X since X is the whole space.) It follows that $A = X \cap A = \bigcup_{i=1}^m U_{x_i} \cap A \subseteq \bigcup_{i=1}^m \{x_i\}$. So A has at most m points, and is therefore finite.

Consequently, any infinite subset A of X must have an accumulation point in X.

7. Let X be a topological space and A a subspace of X. Prove that a set $B \subseteq A$ is compact in X if and only if B is compact in A (with respect to the subspace topology on A).

Solution.

Suppose that A is compact in X. Let $\mathcal{C}$ be a covering of B by the subsets of A which are open in the subspace topology on A. Then $\mathcal{C} = (V_i \cap A)_{i\in I}$, where each V_i is an open subset of X (and I is an indexing set). Thus $B \subseteq \bigcup_{i\in I}(V_i \cap A) \subseteq \bigcup_{i\in I} V_i$, so that $\mathcal{D} = (V_i)_{i\in I}$ is a covering of B by the open sets in X. Since A is compact in X there is a finite subset J of I such that $B \subseteq \bigcup_{i\in J} V_i$, and since $B \subseteq A$ it follows that $B = B\cap A \subseteq \bigcup_{i\in J}(V_i \cap A)$, showing that $(V_i \cap A)_{i\in J}$ is a finite subcovering of $\mathcal{C}$. Since $\mathcal{C}$ was arbitrary, this shows that B is compact as a subset of A.

Conversely, suppose that B is compact in the subspace topology, and let $(V_i)_{i\in I}$ be a covering of B by open sets of X. Since $B \subseteq A$ we see that $(V_i \cap A)_{i\in I}$ is a covering of B by subsets of A, and furthermore these subsets are open in the subspace topology. So there is a finite subset J of I such that $(V_i \cap A)_{i\in J}$ is a covering of B, and it follows that $(V_i)_{i\in J}$ is a finite subcovering of the original open covering of B. Thus B is compact in X.

8. Let X be a metric space (or topological space). Prove that X is compact if and only if for every family $(F_i)_{i\in I}$ of closed subsets of X, if $\bigcap_i F_i = \emptyset$ then there is a finite subset $\{i_1, i_2, \dots, i_m\}$ of I such that $F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_m} = \emptyset$.

Solution.

Suppose that X is compact and $(F_i)_{i\in I}$ is a family of closed subsets of X with $\bigcap_{i\in I} F_i = \emptyset$. By De Morgan's Law, $X = X \setminus \emptyset = X \setminus \bigcap_{i\in I} F_i = \bigcup_{i\in I}(X \setminus F_i)$, showing that $(X \setminus F_i)_{i\in I}$ is an open covering of X, since each F_i is closed. Since X is compact, there is a finite subset J of I such that $(X \setminus F_i)_{i\in J}$ is a covering of X. That is, $X = \bigcup_{i\in J}(X \setminus F_i)$. Thus, by De Morgan's Law,

$$\emptyset = X \setminus X = X \setminus \Big(\bigcup_{i\in J}(X \setminus F_i)\Big) = \bigcap_{i\in J}(X \setminus (X \setminus F_i)) = \bigcap_{i\in J} F_i,$$

showing, as desired, that there is a finite subset of I such that the intersection of the corresponding sets F_i is empty.

Conversely, suppose the condition holds. Let $(V_i)_{i\in I}$ be an open covering of X; that is, $X = \bigcup_i V_i$. By De Morgan, $\emptyset = X \setminus X = X \setminus \big(\bigcup_i V_i\big) = \bigcap_i (X \setminus V_i)$. Let $F_i = X \setminus V_i$. Since each V_i is open, each F_i is closed. So $(F_i)_{i\in I}$ is a family of closed sets with empty intersection. By the hypothesis there is a finite subset J of I such that the subfamily $(F_i)_{i\in J}$ also has empty intersection. By De Morgan,

$$X = X \setminus \emptyset = X \setminus \Big(\bigcap_{i\in J} F_i\Big) = X \setminus \Big(\bigcap_{i\in J} X \setminus V_i\Big) = \bigcup_{i\in J}(X \setminus (X \setminus V_i)) = \bigcup_{i\in J} V_i.$$

Hence an arbitrary open covering of X has a finite subcovering, and therefore X is compact.

9. Let X be a metric space (or topological space). Prove that X is compact if and only if every family $(F_i)_{i\in I}$ of closed subsets of X with the property that every finite subfamily $(F_{i_1}, F_{i_2}, \dots, F_{i_m})$ has a non-empty intersection has, itself, a non-empty intersection.

Solution.

This is just the contrapositive of the result proved in the previous question.