THE UNIVERSITY OF SYDNEY
PURE MATHEMATICS 3901

Metric Spaces 2000

Tutorial 8

1. Let (X, d) be the metric space considered in Q.4 of the assignment: X is set of all positive integers, and for all n, $m \in X$,

$$d(n,m) = \begin{cases} 0 & \text{if } n = m \\ \frac{1}{v(|n-m|)} & \text{if } n \neq m, \end{cases}$$

where $v(n)$ is the largest power of 2 that is a factor of n.

(*i*) Determine all $n \in X$ such that $d(n, 43) < 0.001$.

(*ii*) Show that the sequence $(1, 5, 21, \dots)$, where the nth term a_n is given by $a_n = (2^{2n} - 1)/3 = \sum_{i=0}^{n-1} 2^{2i}$, is a Cauchy sequence in X.

(*iii*) Show that the sequence in Part (*ii*) is not convergent.

(*iv*) It is an elementary fact of number theory that every positive integer n can be uniquely written in the form $\sum_{k=0}^{\infty} c_k 2^k$, where each c_k is 0 or 1, and only finitely many of the c_k are nonzero. (The binary notation for n is then $c_r c_{r-1} \dots c_0$, where r is the largest value of k for which $c_k \neq 0$.) Define $\widehat{X}$ to be the set of all formal sums $\sum_{k=1}^{\infty} c_k 2^k$, where each c_k is 0 or 1 (without any other restriction). Show how to define a metric $\widehat{d}$ on $\widehat{X}$ so that (X, d) is a metric subspace of $(\widehat{X}, \widehat{d})$, and $(\widehat{X}, \widehat{d})$ is complete. (The elements of $\widehat{X}$ are called *2-adic integers*, and there is also a natural way to define addition and multiplication on $\widehat{X}$. One can similarly construct p-adic integers for any integer $p > 1$.)

Solution.

(*i*) The least $k \in \mathbb{Z}^+$ such that $1/2^k < 0.001$ is $k = 10$ (as $1/2^9 \approx 0.00195$ and $1/2^{10} \approx 0.000977$). So $d(n, 43) < 0.001$ if and only if $n = 43 + 1024m$ for some nonnegative integer m.

(*ii*) Let $\varepsilon > 0$. Since $1/2^k \to 0$ as $k \to \infty$ we may choose $k \in \mathbb{Z}^+$ so that $1/2^k < \varepsilon$. Now let $n > m > k$. Then

$$|a_m - a_n| = \left| \frac{(2^{2n} - 1) - (2^{2m} - 1)}{3} \right| = 2^{2m} \left(\frac{2^{2n-2m} - 1}{3} \right)$$

and since $(2^{2n-2m} - 1)/3$ is an odd integer it follows that $v(|a_m - a_n|) = 2^{2m}$. Since $2m > k$ we see that $d(a_m, a_n) = \frac{1}{v(|a_m - a_n|)} < 1/2^{2m} < 1/2^k < \varepsilon$. Similarly, if $m > n > k$ then $d(a_m, a_n) = 1/2^{2n} < \varepsilon$, and if $m = n > k$ then $d(a_m, a_n) = 0 < \varepsilon$. So $d(a_m, a_n) < \varepsilon$ whenever m, $n > k$. So (a_n) is a Cauchy sequence.

(*iii*) The metric d can also be described as follows. Let a, b be nonnegative integers, and let $a = \sum_{k=1}^{\infty} s_k 2^k$ and $b = \sum_{k=1}^{\infty} t_k 2^k$ be the binary expansions of a and b (so that the coefficients s_k and t_k are all 0 or 1, and almost all of them are 0. (In mathematical terminology, "almost all" means "all except for possibly a finite number".) If $a \neq b$, let K be the least k such that $s_k \neq t_k$. Then $s_K - t_K = \pm 1$, and $s_k - t_k = 0$ for $k < K$. So

$$a - b = \pm 2^K + \sum_{k>K} (s_k - t_k) 2^k = 2^K \Big(\pm 1 + \sum_{i=1}^{\infty} (s_{K+i} - t_{K+i}) 2^i \Big),$$

and we see that the highest power of 2 that is a divisor of $|a - b|$ is 2^K. So $d(a, b) = 2^{-K}$. We have shown that if $a \neq b$ then $d(a, b) = 2^{-K}$, where K is the least value of k for which the k-th binary coefficients of a and b are different.

We now show that the sequence (a_n) does not converge. Suppose, for a contradiction, that it does converge, and let the limit be l. Writing the the integer l in binary notation gives $l = \sum_{k=0}^{\infty} t_k 2^k$, where the coefficients t_k are all either 0 or 1, and $t_k = 0$ for all $k > K$, where K is some nonnegative integer. Since $a_n \to l$ as $n \to \infty$, there exists an integer N such that $d(a_n, l) < 2^{-K-2}$ for all $n > N$. In particular, let n be any integer such that $n > N$ and $2n - 2 > K$. Now $a_n = \sum_{i=0}^{n-1} 2^{2i} = \sum_{k=0}^{2n-2} s_k 2^k$, where s_k is 1 for k even and 0 for k odd. Since $2n - 2 > K$, we know that if K is even then $K + 2 \leq 2n - 2$ and $s_{K+2} = 1$, while if K is odd then $K + 1 \leq 2n - 2$ and $s_{K+1} = 1$. But $d(a_n, l) = d(\sum_{k=0}^{2n-2} s_k 2^k, \sum_{k=0}^{\infty} t_k 2^k) = 2^{-L}$ where L is the least k such that $s_k \neq t_k$. Since $t_k = 0$ for all $k > K$, we know that either $s_{K+1} \neq t_{K+1}$ (if K is odd) or $s_{K+2} \neq t_{K+2}$ (if K is even). In either case, $L \leq K + 2$. So $d(a_n, l) = 2^{-L} \geq 2^{-K-2}$. But $d(a_n, l) < 2^{-K-2}$, since $n > N$. This contradiction shows that the sequence (a_n) must converge.

(*iv*) Let $x, y \in \widehat{X}$. Then $x = \sum_{i=0}^{\infty} s_k 2^k$ and $y = \sum_{i=0}^{\infty} t_k 2^k$ for some coefficients s_k and t_k (which are all either 0 or 1). If $x \neq y$, let K be the least integer such that $s_k \neq t_k$. Define $\widehat{d}(x, y) = 2^{-K}$. Put $\widehat{d}(x, y) = 0$ if $x = y$. Observe that in the case that almost all of the s_k's and t_k's are zero, so that x and y can be identified with nonnegative integers, $\widehat{d}(x, y) = d(x, y)$.

We need to show that $\widehat{d}$ is a metric on $\widehat{X}$. It is clear that $\widehat{d}(x, y) = \widehat{d}(y, x)$ for all x, $y \in X$, and if $x \neq y$ then $\widehat{d}(x, y) > 0$, and if $x = y$ then $\widehat{d}(x, y) = 0$. It remains to check the triangle inequality. Let x, y, z be arbitrary points of $\widehat{X}$. If $x = y$ or $y = z$ or $x = z$ then it is trivial that $\widehat{d}(x, y) + +\widehat{d}(y, z) \geq \widehat{d}(x, z)$. So assume that x, y and z are all distinct. Let $x = \sum_{k=0}^{\infty} r_k 2^k$, $y = \sum_{k=0}^{\infty} s_k 2^k$ and $z = \sum_{k=0}^{\infty} t_k 2^k$. Then $\widehat{d}(x, z) = 2^{-K_1}$, where K_1 is the least k such that $r_k \neq t_k$. Similarly, $\widehat{d}(x, y) = 2^{-K_2}$ where K_2 is the least k with $r_k \neq s_k$, and $\widehat{d}(y, z) = 2^{-K_3}$ where K_3 is the least k with $s_k \neq t_k$. Now since $r_{K_1} \neq t_{K_1}$, we must have either $r_{K_1} \neq s_{K_1}$ or $s_{K_1} \neq t_{K_1}$ (or both). If $r_{K_1} \neq s_{K_1}$ then the least k with $r_k \neq s_k$ is less than or equal to K_1; that is, $K_2 \leq K_1$. Similarly, if $s_{K_1} \neq t_{K_1}$ then $K_3 \leq K_1$. So we must have either $K_2 \leq K_1$ or $K_3 \leq K_1$. Hence either $2^{-K_2} \geq 2^{-K_1}$ or $2^{-K_3} \geq 2^{-K_1}$, and in either case $2^{-K_2} + 2^{-K_3} > 2^{-K_1}$ (since $2^{-K_2} + 2^{-K_3} > \max\{2^{-K_2}, 2^{-K_3}\}$).

Before proceeding further, let us note the following fact: if $x = \sum_{k=0}^{\infty} s_k 2^k$ and $y = \sum_{k=0}^{\infty} t_k 2^k$ are elements of $\widehat{X}$ that agree in the first k places, in the sense that $s_k = t_k$ for all $k \le K$, then $\widehat{d}(x,y) < 2^{-K}$. Indeed, either $x = y$, in which case $\widehat{d}(x,y) = 0 < 2^{-K}$, or else $\widehat{d}(x,y) = 2^{-L}$ where $L = \min\{\, k \mid s_k \neq t_k \}$. In the latter case we must have $L > K$, since $s_k = t_k$ for all $k \le K$, whence $2^{-L} < 2^{-K}$, so that $\widehat{d}(x,y) < 2^{-K}$ in either case.

Let us show that $\widehat{X}$ is complete. Suppose that $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence in $\widehat{X}$. We show first that for any integer $K \ge 0$ there exists an N such that for all n, $m > N$ the expansions of x_n and x_m agree in the first K places. Indeed, given K, we can put $\varepsilon = 2^{-K}$, and then choose N so that $d(x_n, x_m) < \varepsilon$ for all n, $m > N$. If x_n and x_m do not agree in the first K places then $d(x_n, x_m) = 2^{-M}$, where M is the least k such that $a_k \neq b_k$, and in particular we must have $M \le K$. But since $2^{-M} < \varepsilon = 2^{-K}$ we have that $M > K$, a contradiction. So x_n and x_m do agree in the first k places, as claimed. Now for each k we can define a coefficient c_k as follows: choose an integer N_k such that x_n and x_m agree in the first k places for all n, $m > N_k$, and define c_k to be the k-th coefficient of x_{N_k+1} (which is also the k-th coefficient of x_n for all $n > N_k$). Having defined c_k like this for all positive integers k, define $x = \sum_{k=0}^{\infty} c_k 2^k \in \widehat{X}$.

We can now show that $x_n \to x$ as $n \to \infty$. Let $\varepsilon > 0$, and choose K such that $2^{-K} < \varepsilon$. Put $N = \max\{\, N_k \mid 0 \le k \le K \,\}$, with the N_k as above. Let $n > N$. Then for each $k \in \{1, 2, \ldots, K\}$ we have $n > N_k$, and so c_k is the k-th coefficient of x_n. So for each $k \in \{1, 2, \ldots, K\}$, the k-th coefficient of x_n is the same as the k-th coefficient of x. So x_n and x agree in the first K places, and so $\widehat{d}(x_n, x) < 2^{-K} < \varepsilon$. We have shown this on the assumption that $n > N$; so, we have shown, as required, that for all $\varepsilon > 0$ there exists N such that $\widehat{d}(x_n, x) < \varepsilon$ for all $n > N$.

Although we have now answered all parts of the question, it is interesting to note some extra things. In particular, it is possible to define operations of addition and multiplication on $\widehat{X}$. Changing our notation, write elements of $\widehat{X}$ as sequences rather than formal sums (so that we write $\ldots s_3 s_2 s_1 s_0$ instead of $\sum_{k=0}^{\infty} s_k 2^k$). Now to add the sequences $\ldots s_3 s_2 s_1 s_0$ and $\ldots t_3 t_2 t_1 t_0$, proceed as is normal for addition of numbers in binary notation: first add s_0 and t_0 (giving 0, 1 or 10), and if the answer is 10 "put down the 0 and carry 1", and continue on to add s_1, t_1 and any carrying figure, and so on. In this way we obtain a perfectly satisfactory operation of addition on $\widehat{X}$ which extends the operation of addition on X (the positive integers). Multiplication of positive integers also extends naturally to a multiplication operation on $\widehat{X}$. These operations make $\widehat{X}$ into a ring—indeed, integral domain—known as the ring of 2-adic integers. The units are the elements $\ldots s_3 s_2 s_1 s_0$ with $s_0 = 1$. Notice that $\ldots 111 = -1$ (as can easily be checked by adding $\ldots 111$ and $1 = \ldots 001$ using the procedure outlined above). This also accords with the formula for the sum of an infinite geometric series, since $\ldots 111$ means $\sum_{i=0}^{\infty} 2^i$, and by the formula this is $1/(1-2) = -1$. The formula is applicable, because the series

does converge in this space. In a similar fashion the limit of the sequence (a_n) considered in parts (*ii*) and (*iii*) above can be identified with $-1/3$, since the limit is $\sum_{i=0}^{\infty} 2^{2i} = 1/(1-2^2) = -1/3$. (Alternatively, $(2^{2n}-1)/3 \to -1/3$ as $n \to \infty$, since $2^{2n} \to 0$ as $n \to \infty$ using the 2-adic metric.)

2. Metric spaces (X,d) and (Y,d') are said to be *isometric* if there is a bijective function $f\colon X \to Y$ such that $d(a,b) = d'(f(a), f(b))$ for all a, $b \in X$.

(*i*) Show that $d(a,b) = |\arctan a - \arctan b|$ defines a metric on $\mathbb{R}$ (where $\arctan\colon \mathbb{R} \to [-\pi/2, \pi/2]$ is the inverse of tan).

(*ii*) Let d' be the usual metric on $\mathbb{R}$. Find a subspace of $(\mathbb{R}, d')$ that is isometric to $(\mathbb{R}, d)$.

(*iii*) Show that $(\mathbb{R}, d)$ is not complete.

(*iv*) Show that $(\mathbb{R}, d)$ and $(\mathbb{R}, d')$ are homeomorphic (even though one is complete and the other is not).

(*v*) Describe a complete metric space which has $(\mathbb{R}, d)$ as a dense subspace.

Solution.

Observe that arctan gives a bijective map from $\mathbb{R}$ to the interval $(-\pi/2, \pi/2)$, and $d(a,b) = d'(\arctan a, \arctan b)$ for all a, $b \in \mathbb{R}$. Since d' is a metric on $(-\pi/2, \pi/2)$ it follows that d must be a metric on $\mathbb{R}$, by Exercise 4 of Tutorial 5. Since arctan is bijective and the equation $d(a,b) = d'(\arctan a, \arctan b)$ holds, we see that $(\mathbb{R}, d)$ and $((-\pi/2, \pi/2), d')$ are isometric. Now $((-\pi/2, \pi/2), d')$ is a subspace of the complete space $(\mathbb{R}, d')$, and we proved in lectures that a subspace (S, D) of a complete metric space (X, D) is complete if and only if S is closed as a subset of X. In this case, since $(-\pi/2, \pi/2)$ is not closed as a subspace of $(\mathbb{R}, d')$, it follows that $((-\pi/2, \pi/2), d')$ is not complete. Indeed, it is easy to find a Cauchy sequence of points in $(-\pi/2, \pi/2)$ whose limit in $\mathbb{R}$ is $\pi/2$. For example, the sequence (x_n) given by $x_n = \frac{\pi}{2} - \frac{1}{n}$ will do. Such sequences do not converge in the space $((-\pi/2, \pi/2), d')$. Applying the isometry yields a non-convergent Cauchy sequence in $(\mathbb{R}, d)$. Specifically, (y_n) defined by $y_n = \tan(\frac{\pi}{2} - \frac{1}{n})$ is such a sequence.

The identity mapping from $\mathbb{R}$ to $\mathbb{R}$ is actually a homeomorphism from $(\mathbb{R}, d)$ to $(\mathbb{R}, d')$. (So d and d' are equivalent metrics.) For any $a \in \mathbb{R}$ and any $\varepsilon > 0$ we can find a $\delta > 0$ such that $|\tan(x) - \tan(\arctan a)| < \varepsilon$ whenever $|x - \arctan a| < \delta$ (since tan is continuous at the point $\arctan a$). Putting $x = \arctan y$ we conclude that $|y - a| < \varepsilon$ whenever $|\arctan y - \arctan a| < \delta$. Thus $B_d(a, \delta) \subseteq B_{d'}(a, \varepsilon)$. Similarly, the fact that arctan is continuous at a shows that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|\arctan x - \arctan a| < \varepsilon$ whenever $|x - a| < \delta$; that is, $B_{d'}(a, \delta) \subseteq B_d(a, \varepsilon)$.

The completion of $((-\pi/2, \pi/2), d')$ is $([-\pi/2, \pi/2], d')$, and similarly to complete $(\mathbb{R}, d)$ we need to add just two extra points, which we might as well call $+\infty$ and $-\infty$. The formula for the metric is on this bigger space is still $d(a,b) = |\arctan a - \arctan b|$, where we interpret $\arctan \infty$ as $\pi/2$ and $\arctan(-\infty)$ as $-\pi/2$.