THE UNIVERSITY OF SYDNEY

PURE MATHEMATICS 3901

Metric Spaces 2000

Tutorial 7

1. Let $A = S^1$ be the unit circle in $\mathbb{R}^2$ and $B = [0, 2\pi) \subseteq \mathbb{R}$. Prove that the mapping $f\colon B \to A$ defined by

$$f(x) = (\cos x, \sin x)$$

is a continuous bijection, but that f^{-1} is not continuous.

Solution.

The proof that f is bijective and the description of its inverse are covered in junior and intermediate maths units which form part of the prerequisites for Metric Spaces. Nevertheless, for completeness' sake we provide the details.

By elementary calculus the restriction of the function cos to the interval $[0, \pi]$ is continuous and strictly decreasing, and hence determines a bijection $[0, \pi] \to [\cos \pi, \cos 0] = [-1, 1]$. Let arccos be the inverse function $[-1, 1] \to [0, \pi]$. Observe that every point p on S^1 has the form $p = (x, y)$ with $x^2 + y^2 = 1$, and since this implies that $-1 \le x \le 1$ we may define a function $\mathrm{Arg}\colon S^1 \to [0, 2\pi)$ by

$$\mathrm{Arg}(p) = \begin{cases} \arccos(x) & \text{if } p = (x, y) \text{ with } y \ge 0, \\ 2\pi - \arccos(x) & \text{if } p = (x, y) \text{ with } y < 0. \end{cases}$$

(Note that if $y < 0$ then $x < 1$; so $0 < \arccos(x) \le \pi$, and it follows that $2\pi - \arccos(x) \in [0, 2\pi)$, as required.) We show that Arg is the inverse of f.

If $0 \le x \le \pi$ then $\sin x \ge 0$, and so

$$(\mathrm{Arg} \circ f)(x) = \mathrm{Arg}(\cos x, \sin x) = \arccos(\cos x) = x$$

(since $x \in [0, \pi]$). If $\pi < x < 2\pi$ then $\sin x < 0$, and $\arccos(\cos x) = 2\pi - x$ since $\cos x = \cos(2\pi - x)$ and $2\pi - x \in [0, \pi]$. So

$$(\mathrm{Arg} \circ f)(x) = \mathrm{Arg}(\cos x, \sin x) = 2\pi - \arccos(\cos x) = 2\pi - (2\pi - x) = x.$$

Thus $\mathrm{Arg} \circ f$ is the identity function on $[0, 2\pi)$.

Let $p = (x, y)$ be any point on S^1, and write $\theta = \arccos(x) \in [0, \pi]$, so that $\cos\theta = x$ and $\sin\theta \ge 0$. Since p is on the unit circle, $y = \pm\sqrt{1 - x^2} = \pm \sin\theta$. Thus

$$\begin{aligned}(f \circ \mathrm{Arg})(p) &= \begin{cases} (\cos\theta, \sin\theta) & \text{if } y \ge 0 \\ (\cos(2\pi - \theta), \sin(2\pi - \theta)) & \text{if } y < 0, \end{cases} \\ &= \begin{cases} (\cos\theta, \sin\theta) & \text{if } y \ge 0 \\ (\cos\theta, -\sin\theta) & \text{if } y < 0, \end{cases} \\ &= (x, y) \qquad \text{in either case.}\end{aligned}$$

So $f \circ \mathrm{Arg}$ is the identity function on S^1, and this completes the proof that $\mathrm{Arg} = f^{-1}$. Since f has an inverse, it is bijective.

The function $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $x \mapsto (\cos x, \sin x)$ is continuous since both cos and sin are. Since f is obtained from this function by restricting the domain and codomain, it follows that f is continuous.

Turning at last to the heart of the matter, we need to show that $f^{-1} = \mathrm{Arg}$ is not continuous. So we need to find an open subset U of $[0, 2\pi)$ such that $\mathrm{Arg}^{-1}(U)$ is not open in S^1. Intuitively, the reason Arg is not continuous is this: it tears the circle apart at $p = (1, 0)$. The open set U that we seek should therefore be a neighbourhood of $\mathrm{Arg}((1, 0)) = 0$. If $q = (x, y)$ is a point close to $(1, 0)$ with $y < 0$ then $\mathrm{Arg}(q)$ is close to 2π rather than 0; so we should choose U small enough to avoid some points near 2π. Then $(1, 0)$ will be in $\mathrm{Arg}^{-1}(U)$ but not in $\mathrm{Int}(\mathrm{Arg}^{-1}(U))$. So let us put $U = (-\pi, \pi) \cap [0, 2\pi) = [0, \pi)$, which is an open subset of $[0, 2\pi)$ since $(-\pi, \pi)$ is open in $\mathbb{R}$. If $q \in \mathrm{Arg}^{-1}(U)$ then $q = (\cos\theta, \sin\theta)$ for some $\theta \in [0, \pi)$, and so $q = (x, y)$ with $y \ge 0$. But for any $\delta > 0$ the open ball in S^1 with centre $(1, 0)$ and radius δ contains points (x, y) with $y < 0$. For example, the point $(x, y) = (1 - \frac{\delta^2}{8}, -\frac{\delta}{2}\sqrt{(1 - \frac{\delta^2}{16})})$ is in this ball, since it satisfies $x^2 + y^2 = 1$ and $d((x, y), (1, 0)) = \delta/2$. But it is not in $\mathrm{Arg}^{-1}(U)$; so $(1, 0)$ is not in $\mathrm{Int}(\mathrm{Arg}^{-1}(U))$, and so $\mathrm{Arg}^{-1}(U)$ is not open.

The above perhaps makes the proof look longer than it really is; so here is a reformulation. Let $\varepsilon = \pi > 0$. For all $\delta > 0$ the point $q = (1 - \frac{\delta^2}{8}, -\frac{\delta}{2}\sqrt{(1 - \frac{\delta^2}{16})})$ satisfies $d(q, (1, 0)) < \delta$ but $d(\mathrm{Arg}(q), \mathrm{Arg}((1, 0))) \ge \varepsilon$. So it is not true that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $d(\mathrm{Arg}(q), \mathrm{Arg}((1, 0))) < \varepsilon$ whenever $d(q, (1, 0)) < \delta$.

2. Show that the mapping $f\colon \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x$ is a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}^+$. (A *homeomorphism* from one topological space to another is a bijective function f such that f and f^{-1} are both continuous.)

Solution.

The function $g\colon \mathbb{R}^+ \to \mathbb{R}$ defined by $g(x) = \ln(x)$ is inverse to f, and both f and g are continuous.

3. Show that the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x_1, x_2) = (x_1^2, x_2^2)$$

gives a homeomorphism from the space

$$A = \{\, (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 0,\ x_2 \ge 0 \,\}$$

onto A, but that it is not a homeomorphism from $\mathbb{R}^2$ onto A.

Solution.

Concerning the very last part of the question, once we have established that $x \mapsto f(x)$ gives a bijective function from A to A, we will certainly know that it does not give a bijective function from $\mathbb{R}^2$ to A, since $A \neq \mathbb{R}^2$.

Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, the range of f is contained in A. So we may define h to be the function $A \to A$ which agrees with f on A. Define $g: A \to A$ by $g(x_1, x_2) = (\sqrt{x_1}, \sqrt{x_2})$. Since $\sqrt{x^2} = x$ and $(\sqrt{x})^2 = x$ whenever $x \geq 0$, it is clear that g and h are inverses of each other. The functions $\mathbb{R}^+ \to \mathbb{R}^+$ given by $x \mapsto x^2$ and $x \mapsto \sqrt{x}$ are certainly both continuous, and since both projections $(x_1, x_2) \mapsto x_i$ (for $i \in \{1, 2\}$) are continuous it follows that the composites $(x_1, x_2) \mapsto x_i \mapsto x_i^2$ and $(x_1, x_2) \mapsto x_i \mapsto \sqrt{x_i}$ are continuous functions $A \to \mathbb{R}^+$. So h and g are continuous functions $A \to A = (\mathbb{R}^+ \times \mathbb{R}^+)$, since both have continuous components $A \to \mathbb{R}^+$.

4. Let

$$A = \{ (x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \text{ and } (x_1, x_2, x_3) \neq (0, 0, 1) \}$$

Show that A is homeomorphic to $\mathbb{R}^2$.

Solution.

Let S^2 be the unit sphere

$$S^2 = \{ (x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \},$$

and let $N = (0, 0, 1)$ (the North Pole). Then $A = S^2 \setminus \{N\}$. Identify $\mathbb{R}^2$ with the subset $\{ (x, y, 0) \mid x,\, y \in \mathbb{R} \}$ of $\mathbb{R}^3$. The bijection we shall produce from A to $\mathbb{R}^2$ is the projection that maps each point P of A to that point Q of $\mathbb{R}^2$ such that NPQ is a straight line.

Suppose that $P = (x_1, x_2, x_3)$ is a point on A. Then $x_3 \neq 1$. The points (y_1, y_2, y_3) on the line through N and P are given parametrically by the equations

$$\begin{aligned} y_1 &= tx_1 \\ y_2 &= tx_2 \\ y_3 &= 1 + t(x_3 - 1). \end{aligned}$$

and this line meets $\mathbb{R}^2$ at the point Q for which $y_3 = 0$. This corresponds to the parameter value $t = 1/(1 - x_3)$. This gives $y_1 = x_1/(1 - x_3)$ and $y_2 = x_2/(1 - x_3)$. So our projection map $f: A \to \mathbb{R}^2$ satisfies

$$f(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right).$$

On any subset of $\mathbb{R}^3$ which does not contain points with $x_3 = 1$, the function defined by these formulas is continuous. (The proof of this is not the main issue here; so we omit the details. The relevant facts are these: the projections $(x_1, x_2, x_3) \mapsto x_i$ are continuous; sums, differences, products and (where defined) reciprocals of continuous real valued functions (on any space) are continuous; composites of continous functions are continuous; and a function whose codomain is a Cartesian product $X \times Y$ is continuous if both its components are continuous.)

Now suppose that $(y_1, y_2) \in \mathbb{R}^2$. The points on the line joining $Q = (y_1, y_2, 0)$ and N are given parametrically by

$$\begin{aligned} x_1 &= ty_1 \\ x_2 &= ty_2 \\ x_3 &= 1 - t. \end{aligned}$$

This meets the sphere when $x_1^2 + x_2^2 + x_3^2 = 1$; that is, $t^2(y_1^2 + y_2^2) + (1 - t)^2 = 1$, or $t(-2 + (1 + y_1^2 + y_2^2)t) = 0$. The solution $t = 0$ gives the point N, and since $1 + y_1^2 + y_2^2 \neq 0$ there is always a second solution, $t = 2/(1 + y_1^2 + y_2^2)$, giving a point P on A whose coordinates are

$$(x_1, x_2, x_3) = \left(\frac{2y_1}{1 + y_1^2 + y_2^2}, \frac{2y_2}{1 + y_1^2 + y_2^2}, \frac{y_1^2 + y_2^2 - 1}{1 + y_1^2 + y_2^2} \right).$$

The mapping $g: \mathbb{R}^2 \to A$ given by $g(y_1, y_2) = (x_1, x_2, x_3)$ as defined by this formula is continuous for reasons similar to those applicable to the function f above. The construction makes it clear that f and g are inverses of each other, but one can also directly substitute into the formulas to show that $f(g(y_1, y_2)) = (y_1, y_2)$ and $g(f(x_1, x_2, x_3)) = (x_1, x_2, x_3)$ for all $(y_1, y_2) \in \mathbb{R}^2$ and $(x_1, x_2, x_3) \in A$. Thus,

$$\begin{aligned} f(g(y_1, y_2)) &= f(ty_1, ty_2, 1 - t) \qquad \text{where } t = t = 2/(1 + y_1^2 + y_2^2), \\ &= \left(\frac{ty_1}{1 - (1 - t)}, \frac{ty_2}{1 - (1 - t)} \right) = (y_1, y_2) \end{aligned}$$

and similarly

$$\begin{aligned} g(f(x_1, x_2, x_3)) &= g(ux_1, ux_2) \qquad \text{where } u = 1/(1 - x_3), \\ &= (tux_1, tux_2, 1 - t) \end{aligned}$$

where we have

$$\begin{aligned} t = 2/(1 + (ux_1)^2 + (ux_2)^2) &= 2/(1 + u^2(1 - x_3^2) = 2\Big/\left(1 + \frac{1 + x_3}{1 - x_3}\right) \\ &= 2(1 - x_3)/((1 - x_3) + (1 + x_3)) = 1 - x_3. \end{aligned}$$

Thus $tu = 1$ and $1 - t = x_3$; so $g(f(x_1, x_2, x_3)) = (x_1, x_2, x_3)$, as required.

We have now shown that f and g are both continuous and inverse to each other; so they are homeomorphisms $A \to \mathbb{R}^2$ and $\mathbb{R}^2 \to A$. So A is homeomorphic to $\mathbb{R}^2$.

5. Let $X = \mathbb{R}^n$ and let d_1 and d_2 be the metrics on X given by

$$d_1(x, y) = \Bigl(\sum_{k=1}^{n} |x_k - y_k|^p\Bigr)^{1/p},$$

where $p \geq 1$, and

$$d_2(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|).$$

Prove that d_1 and d_2 are equivalent over X. (n.b. Metrics d_1 and d_2 on the same set X are said to be *equivalent* if the open sets of the metric space (X, d_1) are also open sets of the metric space (X, d_2), and vice versa. This is the same as saying that the identity map from X to X is a homeomorphism from (X, d_1) to (X, d_2).)

Solution.

For all $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in R^n we have, for all $i \in \{1, 2, \ldots, n\}$,

$$d_1(x, y) = \Bigl(\sum_{k=1}^{n} |x_k - y_k|^p\Bigr)^{1/p} \geq (|x_i - y_i|^p)^{1/p} = |x_i - y_i|,$$

and therefore $d_1(x, y) \geq \max_i |x_i - y_i| = d_2(x, y)$. And for all k we have $|x_k - y_k| \leq d_2(x, y)$, whence $\sum_{k=1}^{n} |x_k - y_k|^p \leq n(d_2(x, y))^p$, giving

$$d_1(x, y) \leq (n(d_2(x, y))^p)^{1/p} = n^{1/p} d_2(x, y).$$

Thus for all $\varepsilon > 0$ there exists a $\delta > 0$, namely $\delta = \varepsilon$, such that $d_2(x, y) < \varepsilon$ whenever $d_1(x, y) < \delta$, and for all $\varepsilon > 0$ there exists a $\delta > 0$, namely $\delta = \varepsilon / \sqrt[p]{n}$, such that $d_1(x, y) < \varepsilon$ whenever $d_2(x, y) < \delta$. This shows that the identity function from $\mathbb{R}$ with the metric d_1 to $\mathbb{R}$ with the metric d_2 is uniformly continuous, and so is the identity function from $\mathbb{R}$ with d_2 to $\mathbb{R}$ with d_1. In particular, these identity functions are continuous, and so the metrics are equivalent.

6. Let (X, d) be any metric space. Let d_1 and d_2 be the metrics on X defined by

$$d_1(x, y) = \min(1, d(x, y)),$$

and

$$d_2(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that d, d_1 and d_2 are all equivalent on X.

Solution.

Clearly $d_1(x, y) \leq d(x, y)$ for all $x, y \in X$; so the identity mapping from (X, d) to (X, d_1) is (uniformly) continuous. Observe also that for all $a \in \mathbb{R}^+$

we have both $a/(1 + a) < (1 + a)/(1 + a) = 1$ and $a/(1 + a) < a/1 = a$; so $a/(1 + a) < \min(1, a)$, and it follows that

$$d_2(x, y) = \frac{d(x, y)}{1 + d(x, y)} < \min(1, d(x, y)) = d_1(x, y)$$

for all $x, y \in X$. So the identity mapping from (X, d_1) to (X, d_2) is also (uniformly) continuous.

To complete the proof we show that the identity mapping from (X, d_2) to (X, d) is continuous (again, as it happens, uniformly). Given $\varepsilon > 0$, set $\delta = \varepsilon/(1 + \varepsilon)$. If $x, y \in X$ are such that $d_2(x, y) < \delta$ then we have

$$\frac{d(x, y)}{1 + d(x, y)} < \frac{\varepsilon}{1 + \varepsilon},$$

whence $d(x, y)(1 + \varepsilon) < \varepsilon(1 + d(x, y))$, which gives $d(x, y) < \varepsilon$.

Thus all the identity mappings between the spaces (X, d), (X, d_1) and (X, d_2) are continuous (since they are all composites of maps we have done explicitly), and so d, d_1 and d_2 are all (uniformly) equivalent on X.