The University of Sydney
Pure Mathematics 3901

Metric Spaces 2000

Tutorial 6

1. Recall that if X and Y are topological spaces then $f\colon X \to Y$ is continuous at $a \in X$ if and only if $a \in \operatorname{Int}(f^{-1}(U))$ whenever U is an open neighbourhood of $f(a)$. Use this criterion to show that is $f\colon \mathbb{R} \to \mathbb{R}$ is the mapping defined by

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2, \\ 0 & \text{if } x = 2, \end{cases}$$

then f is not continuous at $x = 2$.

Solution.

The given definition of f is peculiar. It is equivalent and simpler to say that $f(x) = x + 2$ for $x \neq 2$ and $f(2) = 0$.

Let $U = (-1, 1)$, which is an open neighbourhood of $0 = f(2)$. If $x \neq 2$ and $x \in f^{-1}(U)$ then $f(x) = x + 2 \in U = (-1, +1)$, and thus $x \in (-3, -1)$. Thus $f^{-1}(U) \subseteq \{2\} \cup (-3, -1)$. There is no $\varepsilon > 0$ such that the open ball $B(2, \varepsilon) = (2 - \varepsilon, 2 + \varepsilon)$ is contained in $f^{-1}(U) \subseteq \{2\} \cup (-3, -1)$; so $2 \notin \operatorname{Int}(f^{-1}(U)$. Thus f is not continuous at 2, since there is an open neighbourhood U of $f(2)$ with $2 \notin \operatorname{Int}(f^{-1}(U)$.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the mapping defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Prove that f is not continuous at the point $x = 0$.

Solution.

Let $U = (0, 2)$, an open neighbourhood of $1 = f(0)$. Since $f(x)$ only ever takes the two values 1 and -1, we see that

$$f^{-1}(U) = \{\, x \mid f(x) \in (0, 2) \,\} = \{\, x \mid f(x) = 1 \,\} = [0, \infty).$$

Since $0 \notin \operatorname{Int}([0, \infty))$ it follows that f is not continuous at 0.

3. Let (X, d_X) and (Y, d_Y) be two metric spaces and let b be a fixed but arbitrary point in Y. Let $f\colon X \to Y$ be a mapping defined by $f(x) = b$ for all $x \in X$. Prove that f is a continuous mapping on X.

Solution.

Let U be an open subset of Y. If $b \in U$ then $f(x) \in U$ for all $x \in X$, and so $f^{-1}(U) = X$. If $b \notin U$ then there is no $x \in X$ with $f(x) \in U$, and so $f^{-1}(U) = \emptyset$. In either case, $f^{-1}(U)$ is an open subset of X. Thus the preimage of every open subset of Y is an open subset of X; thus f is a continuous function from X to Y. (Note that the proof applies equally well if X and Y are any topological spaces.)

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the mapping defined by

$$f(x) = \begin{cases} x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Show that f is not continuous at $x = 0$, and hence show that f is not a continuous mapping on $\mathbb{R}$.

Solution.

$f^{-1}(1/2, 3/2) = (1/2, 3/2) \cup \{0\}$, and $0 \notin \operatorname{Int}((1/2, 3/2) \cup \{0\})$. So f is not continuous at 0. So it is not true that f is continuous at all points of $\mathbb{R}$.

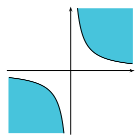
5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = xy$. Determine the sets

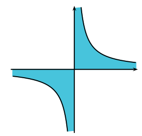
$$S_1 = f^{-1}\big((1, \infty)\big) \qquad \text{and} \qquad S_2 = f^{-1}\big((0, 1)\big).$$

Draw the sets S_1 and S_2 in $\mathbb{R}^2$.

Solution.

The regions in question lie in the 1st and 3rd quadrants, and are unbounded. In each case the hyperbola $xy = 1$ is part of the frontier, but not in the set itself. For S_2 the coordinate axes are part of the frontier but not in the set.





Note that since $(1, \infty)$ and $(0, 1)$ are open subsets of $\mathbb{R}$, and the function f is continuous, the sets S_1 and S_2 are open subsets of $\mathbb{R}^2$.

6. Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that a mapping $f\colon X \to Y$ is continuous on X if and only if for all closed sets F in Y, the set $f^{-1}(F)$ is closed in X.

Solution.

(This is valid for general topological spaces, not just metric spaces.) Write $\mathcal{O}_X$ for the collection of all open subsets of X and $\mathcal{O}_Y$ for the collection of all open subsets of Y. Put $\mathcal{C}_X = \{ X \setminus O \mid O \in \mathcal{O}_X \}$, the collection of closed subsets of X and $\mathcal{C}_Y = \{ Y \setminus O \mid O \in \mathcal{O}_Y \}$, the collection of closed subsets of Y. By definition, f is continuous at $a \in X$ if and only if for all $U \in \mathcal{O}_Y$, if $f(a) \in U$ then $a \in \text{Int } (f^{-1}(U))$. So f is continuous at all points if X if and only if for all $U \in \mathcal{O}_Y$ and all $a \in X$, if $a \in f^{-1}(U)$ then $a \in \text{Int } (f^{-1}(U)$. That is, f is continuous on X if and only if for all $U \in \mathcal{O}_Y$, all points of $f^{-1}(U)$ are interior points. This holds if and only if $f^{-1}(U) \in \mathcal{O}_X$.

Suppose that f is continuous on X and let $F \in \mathcal{C}_Y$. Then $Y \setminus F \in \mathcal{O}_Y$, and so $f^{-1}(Y \setminus F) \in \mathcal{O}_X$. But $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$; so $X \setminus f^{-1}(F) \in \mathcal{O}_X$, and so $f^{-1}(F) \in \mathcal{C}_X$.

Conversely, suppose that $f^{-1}(F) \in \mathcal{C}_X$ for all $F \in \mathcal{C}_Y$, and let $U \in \mathcal{O}_Y$. Then $Y \setminus U \in \mathcal{C}_Y$; so $X \setminus f^{-1}(U) = f^{-1}(Y \setminus U) \in \mathcal{C}_X$, and so $f^{-1}(U) \in \mathcal{O}_X$. This holds for all $U \in \mathcal{O}_Y$; so f is continuous on X.

7. Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that a mapping $f\colon X \to Y$ is continuous on X if and only if for all subsets A of X,

$$f(\overline{A}) \subseteq \overline{f(A)}.$$

Solution.

Note that if $S \subseteq X$ and $T \subseteq Y$ then $f(S) \subseteq T$ if and only if $S \subseteq f^{-1}(T)$. Now if f is continuous and $A \subseteq X$ we have $f(A) \subseteq \overline{f(A)}$, and so $A \subseteq f^{-1}(\overline{f(A)})$. By Question 6, since $\overline{f}(A)$ is closed, so is $f^{-1}(\overline{f(A)})$. So $A \subseteq f^{-1}(\overline{f(A)})$ gives $\overline{A} \subseteq f^{-1}(\overline{f(A)})$, and so $f(\overline{A}) \subseteq \overline{f(A)}$.

Conversely, suppose that $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$, and let $F \in \mathcal{C}_Y$. Put $A = f^{-1}(F)$. Then $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. But $f(A) \subseteq F$ and F is closed; so $\overline{f(A)} \subseteq F$, and therefore $\overline{A} \subseteq f^{-1}(F) = A$, which shows that A is closed. So the preimages of all closed sets are closed, and thus f is continuous (by Question 6).

8. Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that a mapping $f\colon X \to Y$ is continuous on X if and only if for all subsets B of Y,

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}).$$

Solution.

Suppose f is continuous and let $B \subseteq Y$. Put $A = f^{-1}(B)$. Then $f(A) \subseteq B$, and by Question 7, $\overline{A} \subseteq f^{-1}(\overline{f(A)}) \subseteq f^{-1}(\overline{B})$. That is, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Conversely, suppose that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \subseteq Y$, and let $A \subseteq X$. Put $B = f(A)$. Then $A \subseteq f^{-1}(B)$; so $\overline{A} \subseteq \overline{f^{-1}(B)}$, and so (by our hypothesis) $\overline{A} \subseteq f^{-1}(\overline{B})$. Hence $f(\overline{A}) \subseteq \overline{B} = \overline{f(A)}$, and as this holds for all $A \subseteq X$ it follows from Question 7 that f is continuous.

9. Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that a mapping $f\colon X \to Y$ is continuous on X if and only if for all subsets B of Y,

$$f^{-1}(\text{Int } B) \subseteq \text{Int } f^{-1}(B).$$

Solution.

This follows readily from the fact, proved in the solution to Question 6 above, that f is continuous if and only if the preimage of every open set is open, and this proof was given in lectures. It is also easy to show that the condition in this question is equivalent to the one given in Question 8.

Suppose that f is continuous and let $B \subseteq Y$. Put $D = Y \setminus B$. Then $\overline{D} = Y \setminus (\text{Int } (B))$; so by Question 8,

$$\overline{f^{-1}(D)} \subseteq f^{-1}(Y \setminus (\text{Int } (B))) = X \setminus f^{-1}(\text{Int } (B)).$$

Taking complements gives $f^{-1}(\text{Int } (B)) \subseteq X \setminus \overline{f^{-1}(D)} = \text{Int } (X \setminus (f^{-1}(D)))$. But $X \setminus (f^{-1}(D) = f^{-1}(Y \setminus D) = f^{-1}(B)$, and so we have shown that $f^{-1}(\text{Int } (B)) \subseteq \text{Int } (f^{-1}(B)$.

Conversely, suppose that $f^{-1}(\text{Int } (B)) \subseteq \text{Int } (f^{-1}(B)$ for all $B \subseteq Y$. Let $D \subseteq Y$ and put $B = Y \setminus D$. Then $\text{Int } B = Y \setminus \overline{D}$; so

$$X \setminus f^{-1}(\overline{D}) = f^{-1}(Y \setminus \overline{D}) \subseteq \text{Int } (f^{-1}(B)),$$

and taking complements gives $\overline{X \setminus f^{-1}(B)} = X \setminus \text{Int } (f^{-1}(B)) \subseteq f^{-1}(\overline{D})$. But $X \setminus (f^{-1}(B) = f^{-1}(Y \setminus B) = f^{-1}(D)$, and so we have shown that $\overline{f^{-1}(D)} \subseteq f^{-1}(\overline{D})$ for all $D \subseteq Y$. By Question 8, f is continuous.

10. Give an example of a mapping $f\colon X \to Y$, where X and Y are metric spaces, such that f is continuous and closed, but not open.

Solution.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ for all x. Then f is continuous. However, f is not open. We show this by finding an open subset U of $\mathbb{R}$ such that $f(U)$ is not an open subset of $\mathbb{R}$. Let $U = (-1, 1)$. (In fact, any nonempty open subset of $\mathbb{R}$ would do equally well.) Then $f(U) = \{0\}$, which is not open (since 0 is in the set but there is no $\varepsilon > 0$ such that $(0 - \varepsilon, 0 + \varepsilon$ is contained in the set). If C is any closed subset of $\mathbb{R}$ then $f(C) = \{0\}$ if $C \neq \emptyset$, while $f(\emptyset = \emptyset)$. In either case $f(C)$ is closed. So f takes closed sets to closed sets, and so f is closed.

11. Give an example of a continuous mapping $f: \mathbb{R} \to \mathbb{R}$, such that f is neither open nor closed.

Solution.

Let f be defined by $f(x) = e^{-x^2}$ for all $x \in \mathbb{R}$. Observe that f takes only positive values, has a maximum of 1 at $x = 0$, and approaches 0 as $x \to \infty$ and as $x \to -\infty$. So $f(\mathbb{R}) = (0, 1]$. As $\mathbb{R}$ is both open and closed, while $(0, 1]$ is neither open nor closed, f is neither an open mapping nor a closed mapping (although it is obviously continuous).

12. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Show that f is not open.

Solution.

Let $U = \mathbb{R}$. Then U is an open subset of $\mathbb{R}$, but $f(U) = \{\, x^2 \mid x \in \mathbb{R} \,\} = [0, \infty)$ is not open. So f is not an open mapping. (It is true that f is a closed mapping, however. For suppose that C is any closed subset of $\mathbb{R}$, and let t be a point in the closure of $f(C)$. Then there exists a sequence $(a_n)_{n=0}^{\infty}$ of points in $f(C)$ with $\lim_{n\to\infty} a_n = t$. Since $a_n \in f(C)$ for all n there exists $x_n \in C$ with $a_n = f(x_n) = x_n^2$. Since the sequence (a_n) is convergent it is bounded; so there exists $B \in \mathbb{R}$ with $|x_n|^2 = |a_n| < B$ for all n. So $|x_n| < \sqrt{B}$ for all n, and so the sequence (x_n) is bounded too. Hence $\limsup_{n\to\infty} x_n$ exists. Write $l = \limsup n \to \infty x_n$. For each $\varepsilon > 0$ there is an n such that $|l - x_n| < \varepsilon$; so for each $k \in \mathbb{Z}^+$ we may choose n_k such that $|l - x_{n_k}| < 1/k$. Then $\lim_{k\to\infty} x_{n_k} = l$. But each $x_{n_k} \in C$; so $l \in \overline{C}$. But C is closed; so $l \in C$. Furthermore,

$$l^2 = \lim_{k\to\infty} x_{n_k}^2 = \lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} a_n = t.$$

Thus $t = f(l) \in f(C)$, and we have shown that every point of $\overline{f(C)}$ is in $f(C)$. That is, $f(C)$ is closed. As this holds for all closed subsets C of $\mathbb{R}$, the mapping f is closed.

13. Let X and Y be any sets and $f: X \to Y$ a mapping. Let X_1 be a subset of X and $f_{X_1}: X_1 \to Y$ the restriction of f to X_1 (defined by $f_{X_1}(x) = f(x)$ for all $x \in X_1$).

Prove that for all subsets B of Y, $f_{X_1}^{-1}(B) = f^{-1}(B) \cap X_1$.

Solution.

$$\begin{aligned} f_{X_1}^{-1}(B) &= \{\, x \in X_1 \mid f_{X_1}(x) \in B \,\} \\ &= \{\, x \in X \mid x \in X_1 \text{ and } f(x) \in B \,\} \\ &= X_1 \cap \{\, x \in X \mid f(x) \in B \,\} = f^{-1}(B) \cap X_1. \end{aligned}$$

14. Let X, Y and Z be any sets and let $f: X \to Y$ and $g: Y \to Z$ be any mappings. Let $g \circ f: X \to Z$ be the composite mapping defined by

$$(g \circ f)(x) = g\big(f(x)\big) \qquad \text{(for all } x \in X.)$$

Prove that for any subset $B \subseteq Z$, $(g \circ f)^{-1}(B) = f^{-1}\big(g^{-1}(B)\big)$.

Deduce that if X, Y and Z are topological spaces and f and g are continuous then $g \circ f$ is continuous.

Solution.

$$\begin{aligned} (g \circ f)^{-1}(B) &= \{\, x \in X \mid (g \circ f)(x) \in B \,\} \\ &= \{\, x \in X \mid g(f(x)) \in B \,\} \\ &= \{\, x \in X \mid f(x) \in g^{-1}(B) \,\} \\ &= f^{-1}(g^{-1}(B)). \end{aligned}$$

Let B be any open subset of Z. If g is continuous then $g^{-1}(B)$ is open in Y. If f is also continuous then $f^{-1}(g^{-1}(U))$ is open in X. Thus if both f and g are continuous then $(g \circ f)^{-1}(U)$ is open whenever U is open, and thus $g \circ f$ is continuous.