THE UNIVERSITY OF SYDNEY

PURE MATHEMATICS 3901

Metric Spaces 2000

Tutorial 4

1. If a sequence $(x_n)_{n=1}^{\infty}$ in a metric space X is convergent and has limit x, show that every subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) is convergent and has the same limit x.

Solution.

Let $\varepsilon > 0$. Since $x_n \to x$ as $n \to \infty$, there is N such that $d(x_n, x) < \varepsilon$ for all $n > N$. By the definition of the concept of a subsequence, $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers. (That is, $n_1 < n_2 < n_3 < \cdots$.) So there exists K such that $n_k > N$ for all $k > K$. Thus $d(x_{n_k}, x) < \varepsilon$ for all $k > K$. As ε was arbitrary, this shows that $\lim_{k\to\infty} x_{n_k} = x$.

Alternatively, if one is prepared to use the corresponding result for sequences of numbers, which is presumably covered in any course on sequences and series, one can argue as follows. Since $x_n \to x$ as $n \to \infty$ it follows that $d(x_n, x) \to 0$ as $n \to \infty$. Since $(x_{n_k})_{k=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$, so also $(d(x_{n_k}, x))_{k=1}^{\infty}$ is a subsequence of $(d(x_n, x))_{n=1}^{\infty}$. Any subsequence of a convergent sequence of real numbers converges to the same limit as the sequence itself; so $d(x_{n_k}, x) \to 0$ as $k \to \infty$. That is, $x_{n_k} \to x$ as $k \to \infty$.

2. Recall that ℓ^{∞} is the metric space of all bounded sequences in $\mathbb{R}$, with metric d given by

$$d(x, y) = \sup_{k\in\mathbb{N}} |x_k - y_k|.$$

Let M be the subset consisting of all sequences $x = (x_k)$ with at most finitely many nonzero terms. Show that M is not closed. [Hint: Try to produce a sequence in M converging to a point not in M.]

Solution.

For $n = 1, 2, \ldots$, let x_n be the sequence $(x_{n,i})_{i=1}^{\infty}$ defined by

$$x_{n,i} = \begin{cases} 1/i & \text{if } 1 \le i \le n \\ 0 & \text{if } i > n. \end{cases}$$

That is, $x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots))$. Then (x_n) is a sequence in M. Let x be the sequence whose i-th term is $1/i$, for all i. That is, $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots,)$. Then $x \in \ell^{\infty}$ but $x \notin M$. Moreover, for all $n \in \mathbb{Z}^+$,

$$d(x^{(n)}, x) = \sup\{\tfrac{1}{n+1}, \tfrac{1}{n+2}, \tfrac{1}{n+3}, \ldots\} = \tfrac{1}{n+1}.$$

Thus $d(x_n, x) \to 0$ as $n \to \infty$, which means that $(x^{(n)}) \to x$ as $n \to \infty$. Hence x is in the closure of M, since it is the limit of a sequence in M. Since $x \notin M$ it follows that M is not equal to its closure; that is, M is not closed.

3. Let $X = \mathcal{C}[0, 1]$, the set of continuous functions on $[0, 1]$, and let d be the metric on X defined by

$$d(f, g) = \int_0^1 |f(x) - g(x)|\, dx.$$

For each $n \in \mathbb{Z}^+$ define $f_n \in X$ by $f_n(x) = x^n$ for all $x \in [0, 1]$.

(*i*) Show that the sequence $(f_n)_{n=1}^{\infty}$ converges in X, and find its limit f.

(*ii*) Show that the function f in Part (*i*) is not the pointwise limit of the sequence (f_n).

[Hint: f is the continuous function which agrees with the pointwise limit for almost all $x \in [0, 1]$.]

Solution.

(*i*) Let f be the zero function, $f(x) = 0$ for all $x \in [0, 1]$. Then $f \in X$, and for all $n \in \mathbb{Z}^+$

$$d(f_n, f) = \int_0^1 |f_n(x) - f(x)|\, dx = \int_0^1 x^n\, dx = 1/(n + 1),$$

which tends to 0 as $n \to \infty$. So (f_n) converges to f.

(*ii*) It is not true that $\lim_{n\to\infty} f_n(x) = f(x)$ holds for all $x \in [0, 1]$, since $f_n(1) = 1^n = 1$ for all $n \in \mathbb{Z}^+$, giving $f_n(1) \to 1$ as $n \to \infty$, whereas $f(1) = 0$. So f is not the pointwise limit of f_n.

4. Let d be the metric on $X = \mathcal{C}[a,b]$ defined by

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

Let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{C}[a,b]$, and suppose that (f_n) converges uniformly on $[a,b]$ to some function f. Prove that f is continuous on $[a,b]$, and hence show that (f_n) converges in (X,d).

Solution.

We must show that f is continuous at each x_0 in $[a,b]$. So we must show that for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever x satisfies $|x - x_0| < \delta$.

Let $\varepsilon > 0$. Since $f_n \to f$, there is N such that $d(f_n, f) < \varepsilon/3$ for all $n \geq N$. For all $x \in [0,1]$,

$$|f_n(x) - f(x)| \leq \sup_{t\in[0,1]} |f_n(t) - f(t)| = d(f_n, f);$$

so $|f_n(x) - f(x)| < \varepsilon/3$ for all $n \geq N$ and all $x \in [a,b]$. Now since f_N is continuous at x_0, there exists $\delta > 0$ such that $|f_N(x) - f_N(x_0)| < \varepsilon/3$ whenever $|x - x_0| < \delta$. Hence if x satisfies $|x - x_0| < \delta$, then

$$\begin{aligned}|f(x) - f(x_0)| &< |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Hence f is continuous on $[a,b]$.

5. Let (X,d) be as in Question 4, and suppose that (f_n) is a convergent sequence in this space with limit f. (In other words, (f_n) converges to f uniformly on $[a,b]$.)

Prove that $\displaystyle\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx \quad$ as $n \to \infty$.

Solution.

Let $n \in \mathbb{Z}^+$. Since

$$-|f_n(x) - f(x)| \leq f_n(x) - f(x) \leq |f_n(x) - f(x)|$$

for all $x \in [a,b]$, it follows that

$$-\int_a^b |f_n(x) - f(x)|\,dx \leq \int_a^b f_n(x) - f(x)\,dx \leq \int_a^b |f_n(x) - f(x)|\,dx,$$

and therefore

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| = \left|\int_a^b f_n(x) - f(x)\,dx\right| \leq \int_a^b |f_n(x) - f(x)|\,dx.$$

Note that for all $x \in [a,b]$,

$$|f_n(x) - f(x)| \leq \sup_{t\in[0,1]} |f_n(t) - f(t)| = d(f_n, f).$$

So we have

$$\begin{aligned}\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| &\leq \int_a^b |f_n(x) - f(x)|\,dx \\ &\leq \int_a^b d(f_n, f)\,dx \\ &\leq (b-a)d(f_n, f) \\ &\longrightarrow 0 \qquad \text{as } n \to \infty.\end{aligned}$$

Hence $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ as $n \to \infty$.