THE UNIVERSITY OF SYDNEY
PURE MATHEMATICS 3901

Metric Spaces 2000

Tutorial 3

1. Sketch (where possible) the following sets A, and decide whether A is an open subset, or a closed subset, or neither, of the appropriate space $\mathbb{R}^n$. Then for each A, find $\mathrm{Int}(A)$, $\overline{A}$ and $\mathrm{Fr}(A)$.

(*i*) $A = \bigcup_{n\in\mathbb{N}} (n,\ n+1)$ (where $\mathbb{N} = \{0,\ 1,\ 2,\ \ldots\}$).

(*ii*) $A = \{\,(x_1, x_2) \in \mathbb{R}^2 \mid x_1x_2 = 0\,\}$.

(*iii*) $A = \{\,(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \mathbb{Q}\,\}$ (where $\mathbb{Q}$ is the set of rational numbers).

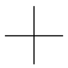
(*iv*) $A = \{\,(x_1, 0) \in \mathbb{R}^2 \mid 0 < x_1 < 4\,\}$.

Solution.

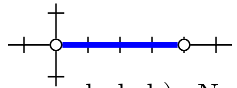
(*i*) The set A is the positive half of the real line with the integers removed:

Since each open interval $(n, n+1)$ is open, the set A is a union of open sets, and hence open. (Note that in $\mathbb{R}^1$ with the usual metric, the open interval (a, b) equals the open ball centred at $(a+b)/2$ with radius $(b-a)/2$.) Since A is open, $\mathrm{Int}(A) = A$. The closure $\overline{A}$ is the set of all nonnegative real numbers (since every open interval centred at a positive real number contains a point in an interval $(n, n+1)$ for some n), and

$$\mathrm{Fr}(A) = \overline{A} \setminus \mathrm{Int}(A) = \{0, 1, 2, \ldots\} = \mathbb{N}.$$

(*ii*) This time A is the set of points which lie on one or other of the coordinate axes.  Any circle whose centre is on one of the axes will contain a point not on either axis; so A has no interior points. That is, $\mathrm{Int}(A) = \emptyset$. On the other hand, the complement of A is open: if $(x, y) \in \mathbb{R}^2 \setminus A$ then $x \neq 0$ and $y \neq 0$, and and the open disc with centre (x, y) and radius $\min(|x|, |y|)$ contains no point on either axis (so that $(x, y) \in \mathrm{Int}(\mathbb{R}^2 \setminus A)$). So A is closed; so $\overline{A} = A$. And $\mathrm{Fr}(A) = \overline{A} \setminus \mathrm{Int}(A) = A$.

(*iii*) I can't draw this set (points whose x-coordinate is rational). It is easily seen that every circle in the plane contains points with rational x-coordinate and points with irrational x-coordinate. So all points of $\mathbb{R}^2$ are in $\overline{A}$ and no points are in $\mathrm{Int}(A)$. So $\mathrm{Int}(A) = \emptyset$ and $\overline{A} = \mathbb{R}^2 = \mathrm{Fr}(A)$.

(*iv*) A is the line segment from (0,0) to (4,0): 
(The endpoints $(0, 0)$ and $(4, 0)$ themselves are excluded.) No circle in the plane is composed entirely of points on this line segment; so $\mathrm{Int}(A) = \emptyset$. The points $(0, 0)$ and $(4, 0)$ are in $\overline{A}$ since any circle centred at either of these points will include points of the line segment A. For every other point $(x, y) \in \mathbb{R}^2$ which is not in A one can find a circle with centre (x, y) and radius small enough that it does not contain any point on the line segment. Specifically, if $y \neq 0$ we can choose the radius to be $|y|/2$, and if $y = 0$ then $x > 4$ or $x < 0$, and we can take the radius to be either $\frac{x-4}{2}$ or $\frac{-x}{2}$ (whichever is positive). So such points (x, y) are not in $\overline{A}$. So $\overline{A}$ is the line segment from $(0, 0)$ to $(0, 4)$ including the endpoints. And since $\mathrm{Int}(A)$ is empty, $\mathrm{Fr}(A) = \overline{A}$.

2. Let A be an open subset of a metric space (X, d) and $a \in A$. Is $A \setminus \{a\}$ open in X?

Solution.

Yes. Note first that $X \setminus \{a\}$ is open, for if $x \in X \setminus \{a\}$ is arbitrary then $B_d(x, \frac{1}{2}d(a, x))$ is contained in $X \setminus \{a\}$ (since $a \notin B_d(x, \frac{1}{2}d(a, x))$). Since $A \setminus \{a\} = A \cap (X \setminus \{a\}$, and the intersection of two open sets is always open, the result follows.

3. Let (X, d) be a metric space, and A, B subsets of X with $A \subseteq B$. Prove that $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$.

Solution.

Let $x \in \mathrm{Int}(A)$ be arbitrary. Then there exists $\varepsilon > 0$ with $B_d(x, \varepsilon) \subseteq A$. Since $A \subseteq B$ it follows that $B_d(x, \varepsilon) \subseteq B$. So $x \in \mathrm{Int}(B)$. This holds for all $x \in \mathrm{Int}(A)$; so $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$.

4. Let (X,d) be a metric space and $A \subseteq X$. Let x be a limit point of A. Prove that every open ball with centre x contains an infinite number of points of A, and use this to show that $(A')' \subseteq A'$.

Solution.

Let x be a limit point (accumulation point) of A, and let $B = B_d(x,t)$ be an open ball with centre x. Suppose that B does not contain an infinite number of points of A. Since x is an accumulation point of A there is at least one point of A in $B \setminus \{x\}$; our assumption says that there are only finitely many such points. So let $a_1, a_2, \ldots, a_k$ be all the points of $(B \setminus \{x\}) \cap A$. Since $a_i \neq x$ for each i, each distance $d(a_i,x)$ is positive. Put $s = \min_i(d(a_i,x))$, the smallest of these k positive numbers. Then $d(a_i,x) \geq s$ for each i, and so $a_i \notin B_d(x,s)$ for each i. But since x is an accumulation point of A there is a point $a \in (B_d(x,s) \setminus \{x\}) \cap A$. Now $0 < d(a,x) < s \leq d(a_1,x) < t$ (since $a_1 \in B_d(x,t)$), and it follows that $a \in (B_d(x,t) \setminus \{x\}) \cap A$. But since $a \neq a_i$ for each i (since $d(x,a) < d(x,a_i)$) this contradicts the fact that $a_1, a_2, \ldots, a_k$ are all the points of $(B_d(x,t) \setminus \{x\}) \cap A$. This contradiction shows that our original assumption that B does not contain infinitely many points of A is false. Since B was an arbitrary open ball centred at x, we have shown that every such ball contains infinitely many points of A.

Let $x \in (A')'$, and let B be an open ball with centre x. Then B contains at least one point of A'; so choose $b \in B \cap A'$. Since $b \in B$ and B is open there exists an open ball B_1 with centre at b and $B_1 \subseteq B$. Since $b \in A'$, every open ball centred at b contains infinitely many points of A. In particular, B_1 contains infinitely many points of A, and since $B_1 \subseteq B$ it follows that B contains infinitely many points of A. So B contains at least one point of A different from x. This holds for all open balls containing x; so x is an accumulation point of A. Thus we have shown that every point of $(A')'$ is in A'; that is, $(A')' \subseteq A'$, as required.

5. Let (X,d) be a metric space.

(*i*) If $A \subseteq B \subseteq X$, prove that $A' \subseteq B'$.

(*ii*) If A and B are subsets of X, prove that $(A \cup B)' = A' \cup B'$.

Solution.

(*i*) Suppose that $A \subseteq B \subseteq X$, and let x be an arbitrary point of A'. Let U be an open neighbourhood of x. Then $(U \setminus \{x\}) \cap A \neq \emptyset$.

But since $A \subseteq B$ it follows that $(U \setminus \{x\}) \cap A \subseteq (U \setminus \{x\}) \cap B$. So $(U \setminus \{x\}) \cap B \neq \emptyset$. This holds for all open sets U with $x \in U$; so $x \in B'$. This is true for all $x \in A'$; so $A' \subseteq B'$.

(*ii*) Since $A \subseteq (A \cup B)$, it follows from (*i*) that $A' \subseteq (A \cup B)'$, and equally $B' \subseteq (A \cup B)'$. So $A' \cup B' \subseteq (A \cup B)'$.

Our strategy now is to show that points which are not in A' and not in B' are not in $(A \cup B)'$ (since this implies that if $x \in (A \cup B)'$ then either $x \in A'$ or $x \in B'$; that is, $(A \cup B)' \subseteq A' \cup B'$.) To say that $x \in A'$ is to say that for every open neighbourhood U of x the set $A \cap U \setminus \{x\}$ is nonempty. So to say that $x \notin A'$ is to say that there exists an open set U containing x such that $A \cap U \setminus \{x\} = \emptyset$. Similarly, if $x \notin B'$ then there is an open set V with $x \in V$ and $B \cap V \setminus \{x\} = \emptyset$. Choose such a U and such a V. Then $U \cap V$ is open and $x \in U \cap V$. Moreover,

$$\begin{aligned}(A \cup B) \cap (U \cap V) \setminus \{x\} &= (A \cap (U \cap V) \setminus \{x\}) \cup (B \cap (U \cap V) \setminus \{x\}) \\ &\subseteq (A \cap U \setminus \{x\}) \cup (B \cap V) \setminus \{x\}) = \emptyset.\end{aligned}$$

So $U \cap V$ is an open neighbourhood of x containing no points of $A \cup B$ different from x. So $x \notin (A \cup B)'$.

6. Let (X,d) be a metric space and A, B be two subsets of X. Prove that:

(*i*) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

(*ii*) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(*iii*) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

Show that equality need not hold in Part (*iii*).

Solution.

(*i*) Recall from lectures that the closure of a set S is a closed set containing S and contained in all the closed sets containing S. Now suppose that $A \subseteq B$. Since $B \subseteq \overline{B}$ we have $A \subseteq \overline{B}$. Since $\overline{B}$ is closed and contains A, it contains $\overline{A}$, as required.

(*ii*) We have $A \subseteq \overline{A} \subseteq \overline{A} \cup \overline{B}$ and $B \subseteq \overline{B} \subseteq \overline{A} \cup \overline{B}$. So $A \cup B \subseteq \overline{A} \cup \overline{B}$. Since the union of two closed sets is always closed, $\overline{A} \cup \overline{B}$ is closed. Since it contains $A \cup B$ it must contain the closure of $A \cup B$. So $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

By the first part and the fact that $A \subseteq A \cup B$ it follows that $\overline{A} \subseteq \overline{A \cup B}$. Similarly, since $B \subseteq A \cup B$ we find that $\overline{B} \subseteq \overline{A \cup B}$.

So $\overline{A}\cup\overline{B} \subseteq \overline{A\cup B}$. Since the reverse inclusion was proved above, $\overline{A}\cup\overline{B} = \overline{A\cup B}$.

(*iii*) By Part (*i*) and $A\cap B \subseteq A$ we have $\overline{A\cap B} \subseteq \overline{A}$; similarly $A\cap B\subseteq B$ gives $\overline{A\cap B}\subseteq\overline{B}$. So $\overline{A\cap B}\subseteq\overline{A}\cap\overline{B}$.

Let $X=\mathbb{R}$ with the usual metric. Let A be the open half-line $(0,\infty)$ and B the open half-line $(-\infty,0)$. Then $A\cap B=\emptyset$, and so $\overline{A\cap B}=\emptyset$. But $\overline{A}=[0,\infty)$ and $\overline{B}=(-\infty,0]$; so $\overline{A}\cap\overline{B}=\{0\}\neq\overline{A\cap B}$.

7. Let $A=\{\,(x_1,x_2)\in\mathbb{R}^2 \mid x_1,\ x_2\in\mathbb{Q}\,\}$ (where $\mathbb{Q}$ is the set of all rational numbers). Show that $\overline{A}=\mathbb{R}^2$. Deduce that $\mathbb{R}^2$ is separable.

Solution.

A countable union of finite sets is countable: if A_1, A_2, A_3, … are finite sets then we can list all the elements of $\bigcup_{i=1}^{\infty} A_i$ by listing the elements of A_1 first, then the elements of A_2, then A_3, and so on. It follows that the set $\mathbb{Z}^+\times\mathbb{Z}^+=\{\,(m,n)\mid m,\ n\in\mathbb{Z}^+\,\}$ is countable: it equals $\bigcup_{i=2}^{\infty} A_i$, where $A_i=\{\,(m,n)\mid m,\ n\in\mathbb{Z}^+ \text{ and } m+n=i\,\}$, a finite set (for each $i\geq 2$). Since $(m,n)\mapsto m/n$ is a surjective map from $\mathbb{Z}^+\times\mathbb{Z}^+$ to $\mathbb{Q}^+$, the set of positive rational numbers, it follows that $\mathbb{Q}^+$ is countable. So $\mathbb{Q}$ is countable, since we can list the elements of $\mathbb{Q}$ in the order q_1, $-q_1$, q_2, $-q_2$, …, where q_i is the i-th term in a listing ogf the elements of $\mathbb{Q}^+$. So we obtain a one to one correspondence between $\mathbb{Z}^+$ and $\mathbb{Q}$, and therefore there is a one to one correspondence between $\mathbb{Z}^+\times\mathbb{Z}^+$ and $\mathbb{Q}\times\mathbb{Q}$. But $\mathbb{Z}^+\times\mathbb{Z}^+$ is countable; so $\mathbb{Q}\times\mathbb{Q}$ is countable. That is, A is countable.

Let $(x,y)\in\mathbb{R}^2$, and let $\varepsilon>0$ Choose a positive integer k with $10^{-k}<\varepsilon/\sqrt{2}$, and let X be the integer part of $10^k x$ and Y the integer part of $10^k y$. (That is, $X\in\mathbb{Z}$ satisfies $X\leq 10^k x<X+1$, and similarly for Y.) Then $(10^{-k}X,10^{-k}Y)\in A$, and

$$\begin{aligned} d((10^{-k}X,10^{-k}Y),(x,y)) &= \sqrt{|10^{-k}X-x|^2+|10^{-k}Y-y|^2} \\ &= 10^{-k}\sqrt{|X-10^k x|^2+|Y-10^k y|^2} \\ &< 10^{-k}\sqrt{2}<\varepsilon. \end{aligned}$$

Thus $B((x,y),\varepsilon)$ contains a point of A, and since this holds for all $\varepsilon>0$ it follows that $(x,y)\in\overline{A}$. But (x,y) was an arbitrary point of $\mathbb{R}^2$; so $\overline{A}=\mathbb{R}^2$. In other words, A is dense in $\mathbb{R}^2$. So $\mathbb{R}^2$ has a countable dense subset (and this is what separable means).

8. Let (Y,d_Y) a metric subspace of a metric space (X,d) and $H\subseteq Y$. Prove that H is closed in Y if and only if there exists a closed subset C in X such that $H=C\cap Y$.

Solution.

Let us prove first that a subset J of Y is open in Y if and only if there is an open subset U of X such that $J=U\cap Y$. For $a\in Y$ and $\varepsilon>0$ let us write $B_Y(a,\varepsilon)=\{\,y\in Y\mid d_Y(a,y)<\varepsilon\,\}$, and observe that $B_Y(a,\varepsilon)=Y\cap B_X(a,\varepsilon)$, where $B_X(a,\varepsilon)=\{\,x\in X\mid d(a,x)<\varepsilon\,\}$.

Suppose first that $J=U\cap Y$, where U is open in X. Let $a\in J$. Then $a\in U$, and so there is an $\varepsilon>0$ such that $B_X(a,\varepsilon)\subseteq U$. So

$$B_Y(a,\varepsilon)=Y\cap B_X(a,\varepsilon)\subseteq Y\cap U=J.$$

This shows that a is an interior point of J in the metric space Y, and since a was an arbitrary point of J it follows that J is open in Y.

Conversely, suppose that J is open in Y. Then every point of J is contained in an open ball contained in J. So J is the union of the sets in the collection $\mathcal{S}=\{\,B_Y(a,\varepsilon)\mid B_Y(a,\varepsilon)\subseteq J\,\}$. Now let $\mathcal{T}=\{\,B_X(a,\varepsilon)\mid B_Y(a,\varepsilon)\in\mathcal{S}\,\}$, and let U be the union of all the sets in the collection $\mathcal{T}$. Then U is open, since it is a union of open balls. And

$$Y\cap U=Y\cap\bigcup_{B\in\mathcal{T}} B=\bigcup_{B\in\mathcal{T}}(Y\cap B)=\bigcup_{D\in\mathcal{S}} D=J$$

since the sets in the collection $\mathcal{S}$ are precisely the intersections with Y of the sets in $\mathcal{T}$. So J is the intersection with Y of an open subset of X.

Observe that $Y\cap C=H$ if and only if $Y\cap(X\setminus C)=Y\setminus H$. Since H is closed in Y if and only if $Y\setminus H$ is open in Y, and C is closed in X if and only if $X\setminus C$ is open in X, the result follows. (If $H=Y\cap C$ with C closed, then $Y\setminus H=Y\cap(X\setminus C)$ is open since $X\setminus C$ is open; so H is closed. Conversely, if H is closed we can find an open U with $Y\cap U=Y\setminus H$, and then $H=Y\cap C$ where $C=X\setminus U$.)

9. Let $\mathbb{Z}^+=\{1,\,2,\,3,\,\ldots\}$, the set of all positive integers, considered as a subspace of the metric space $(\mathbb{R},d)$ (where d is the usual metric). Describe the open sets of $\mathbb{Z}^+$.

Solution.

With this metric, all subsets of $\mathbb{Z}^+$ are open. If $n\in\mathbb{Z}^+$ then the open ball with radius $1/2$ centred at n contains n and no other element of $\mathbb{Z}^+$. So $\{n\}$ is an open set in $\mathbb{Z}^+$. Since every subset of $\mathbb{Z}^+$ is a union of sets of this form, all subsets of $\mathbb{Z}^+$ are open.