THE UNIVERSITY OF SYDNEY

MATH Pure Mathematics 3901

Semester 0, 2002 Lecturer:

Tutorial 2

1. Let X be any non-empty set. Define $d(x,y)$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Show that d is a metric on X.

Solution.

It is immediate that $d(x,y) = d(y,x)$ for all $x,\, y \in X$, with $d(x,y) = 0$ if and only if $x = y$. So it remains to prove that $d(y,z) \le d(x,y) + d(x,z)$ for all $x,\, y,\, z \in X$. Now $d(x,y) + d(x,z) \ge 1$ unless $d(x,y) = d(x,z) = 0$, which only happens if $x = y$ and $x = z$, in which case $d(y,z) = 0$ also, giving $d(y,z) = d(x,y) + d(y,z)$. And when $d(x,y) + d(x,z) \ge 1$ it is also true that $d(y,z) \le d(x,y) + d(x,z)$, since $d(y,z) \le 1$.

2. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, define

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$
$$d'(x,\, y) = \max(|x_1 - y_1|,\, |x_2 - y_2|)$$
$$d''(x,y) = \min(|x_1 - y_1|,\, |x_2 - y_2|).$$

Which of d, d', d'' are metrics on $\mathbb{R}^2$?

Solution.

We showed in lectures that $d_p(x,y) = \sqrt[p]{|x_1 - y_1|^p + |x_2 - y_2|^p}$ is a metric on $\mathbb{R}^2$ for all $p \ge 1$. (In fact, we showed the analogous result for $\mathbb{C}^n$.) The function d defined above is d_1, and is therefore a metric. The main part of the proof is the observation that

$$|y_1 - z_1| + |y_2 - z_2| \le (|x_1 - y_1| + |x_2 - y_2|) + (|x_1 - z_1| + |x_2 - x_2|)$$

for all x_i, y_i and z_i (which follows from $|a+b| \le |a| + |b|$ by putting $a = y_i - x_i$ and $b = x_i - z_i$).

We also proved in lectures that $d_\infty(x,y) = \lim_{p\to\infty} d_p(x,y) = \max_i |x_i - y_i|$. That is, the function d' defined above coincides with d_∞ for $\mathbb{R}^2$. It is also a metric, since for some j,

$$\max_i |y_i - z_i| = |y_j - z_j| \le |y_j - x_j| + |x_j - z_j| \le \max_i |x_i - y_i| + \max_i |x_i - z_i|,$$

the other requirements being obviously satisfied.

The function d'' is not a metric, since (for example) $d''((0,1),(0,0)) = 0$, even though $(0,1) \neq (0,0)$.

3. Let $X = \ell^\infty$, the set of all bounded real sequences, that is all real infinite sequences (x_k) such that $\sup_{k\in\mathbb{N}} |x_k| < \infty$, and for $x,\, y \in X$, define

$$d(x,\, y) = \sup_{k\in\mathbb{N}} |x_k - y_k|.$$

Show that d is a metric on X.

Solution.

Let $x,\, y \in X$. Then $x,\, y$ are bounded sequences, and so there exist $A,\, B \in R$ such that $|x_k| < A$ and $|y_k| < B$ for all $k \in \mathbb{N}$. So $|x_k - y_k| \le |x_k| + |y_k| < A+B$ for all k, and therefore $\sup_k |x_k - y_k|$ exists (since every bounded set of real numbers has a supremum). So d is well-defined. Since $|x_k - y_k| = |y_k - x_k|$ for all k it follows that $d(x,y) = d(y,x)$. If $d(x,y) = 0$ then for all i we have $0 \le |x_i - y_i| \le \sup_k |x_k - y_k| = 0$, and so $x = y$; conversely, clearly $d(x,x) = 0$ for all $x \in X$. So it remains to prove the triangle inequality.

Let $x,\, y,\, z \in X$. For all $i \in \mathbb{N}$ we have

$$|y_i - z_i| \le |y_i - x_i| + |x_i - z_i| \le \sup_k |x_k - y_k| + \sup_k |y_k - z_k| = d(x,y) + d(x,z).$$

So $d(x,y) + d(x,z)$ is an upper bound for the set $\{\, |y_i - z_i| \bigm| i \in \mathbb{N}\,\}$, and it follows that $\sup_i |y_i - z_i| \le d(x,y) + d(x,z)$. That is, $d(y,z) \le d(x,y) + d(x,z)$, as required.

4. Let $\mathcal{C}[a,\, b]$ be the set of all continuous real-valued functions defined on $[a,\, b]$. For $f,\, g \in \mathcal{C}[a,\, b]$ define

$$d_1(f,\, g) = \sup_{x\in[a,\, b]} |f(x) - g(x)|$$
$$d_2(f,\, g) = \int_a^b |f(x) - g(x)|\, dx$$

Show that d_1 and d_2 are metrics on $\mathcal{C}[a,\, b]$.

Solution.

It is a standard theorem of real analysis that a continuous function on a closed interval achieves a maximum value on the interval. So for each pair of elements $f,\, g \in \mathcal{C}[a,b]$ there exists a $t \in [a,b]$ such that $d_1(f,g) = |f(t) - g(t)|$. So if $f,\, g,\, h \in \mathcal{C}[a,b]$, then, for some $t \in [a,b]$,

$$d_1(f,g) = |f(t) - g(t)| \le |f(t) - h(t)| + |h(t) - g(t)| \le d_1(f,h) + d_1(h,g),$$

(since $|f(t) - h(t)| \le \sup_{x\in[a,b]} |f(x) - f(x)| = d_1(f,h)$, etc.). It is clear that $d_1(f,g) = d_1(g,f)$, for all $f,\, g \in \mathcal{C}[a,b]$, since $|f(x) - g(x)| = |g(x) - f(x)|$ for all $x \in [a,b]$. And since $\sup_{x\in[a,b]} |f(x) - g(x)| = 0$ if and only if

$|f(x) - g(x)| = 0$ for all $x \in [a, b]$, we see that $d_1(f, g) = 0$ if and only if $f = g$.

Since $\int_a^b |f(x) - g(x)|\,dx = \int_a^b |g(x) - f(x)|\,dx$, we have $d_2(f, g) = d_2(g, f)$ for all $f, g \in \mathcal{C}[a, b]$. Clearly $d_2(f, f) = \int_a^b 0\,dx = 0$, for all $f \in \mathcal{C}[a, b]$. If $f \neq g$ then there exists $t \in [a, b]$ with $|f(t) - g(t)| = c > 0$, and by continuity $|f(x) - g(x)| \geq c/2$ for all x in some neighbourhood of t. Thus, there exist p, q with $a \leq p < q \leq b$ and $|f(x) - g(x)| \geq c/2$ for all $x \in [p, q]$. Since $|f(x) - g(x)| \geq 0$ for all other points $x \in [a, b]$ it follows that

$$d_2(f, g) = \int_a^b |f(x) - g(x)|\,dx \geq (q - p)c/2 > 0.$$

Thus $d_2(f, g) = 0$ only when $f = g$. And for all $f, g, h \in \mathcal{C}[a, b]$,

$$\begin{aligned} d_2(f, g) &= \int_a^b |f(x) - g(x)|\,dx \\ &\leq \int_a^b |f(x) - h(x)| + |h(x) - g(x)|\,dx \\ &\leq \int_a^b |f(x) - h(x)|\,dx + \int_a^b ||h(x) - g(x)|\,dx \\ &= d_2(f, h) + d_2(h, g) \end{aligned}$$

5. For x and y in $\mathbb{R}$, define

$$d'(x, y) = \sqrt{|x - y|}.$$

Show that d' is a metric on $\mathbb{R}$.

Solution.

It is clear that $d'(x, y) = d'(y, x)$, and $d'(x, y) = 0$ if and only if $x = y$. Let $x, y, z \in \mathbb{R}$. Suppose that $d'(y, z) > d'(x, y) + d'(x, z)$. Since $f(x) = x^2$ is an increasing function on $[0, \infty)$ it follows that $(d'(y, z))^2 > (d'(x, y) + d'(x, z))^2$. That is,

$$|y - z| > (\sqrt{|x - y|} + \sqrt{|x - z|})^2 = |x - y| + |x - z| + 2\sqrt{|x - y||x - z|},$$

but since it is a standard fact that

$$|y - x| + |x - z| \geq |y - z|,$$

it follows that $2\sqrt{|x - y||x - z|} < 0$, which is impossible. So we must have $d'(y, z) \leq d'(x, y) + d'(x, z)$.

6. Let (X, d) be a metric space. Define $d': X \times X \to \mathbb{R}$ by

$$d'(x, y) = \min\big(1, d(x, y)\big).$$

Show that d' is a metric on X.

Solution.

Let $x, y \in X$. Since $d(x, y) = d(y, x) \geq 0$ it follows that

$$d'(y, x) = \min(1, d(y, x)) = \min(1, d(x, y)) = d'(x, y) \geq 0.$$

And if $\min(1, d(x, y)) = 0$ then $d(x, y) = 0$, which gives $x = y$ since d is a metric. So $d'(x, y) = 0$ if and only if $x = y$.

Let $x, y, z \in X$. We must show that $d'(x, y) + d'(x, z) \geq d'(y, z)$. Now $d'(y, z) \leq 1$, and so if either $d'(x, y) = 1$ or $d'(x, z) = 1$ then the desired inequality holds. But if both $d'(x, y) < 1$ and $d'(x, z) < 1$ then

$$d'(x, y) + d'(x, z) = d(x, y) + d(x, z) \geq d(y, z) \geq d'(y, z),$$

as required.

7. Let (X, d) be a metric space. Define $d' : X \times X \to \mathbb{R}$ by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that d' is a metric on X.

Solution.

Since $d(x, y) = d(y, x) \geq 0$, also $d'(x, y) = d'(y, x) \geq 0$. And $d'(x, y) = 0$ if and only if $d(x, y) = 0$; so $d'(x, y) = 0$ if and only if $x = y$. Let $x, y, z \in X$, and put $a = d(y, z)$, $b = d(x, y)$ and $c = d(x, z)$. Then $a \leq b + c$. So by Question 7 of Tutorial 2, $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$. Thus $d'(y, z) \leq d'(x, y) + d'(y, z)$.

8. Let X be the set of all real sequences. For $x = (x_k)$ and $y = (y_k)$ in X, define

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}.$$

Show that d is a metric on X.

Solution.

Since $\frac{1}{2^k} \frac{|x_k - y_k|}{1+|x_k - y_k|} \leq \frac{1}{2^k}$ the series defining $d(x, y)$ converges. It is clear that $d(x, y) = d(y, x) \geq 0$, and $d(x, y) = 0$ only if all terms of the series are 0, which forces $x_k = y_k$ for all k, and so $x = y$. If $x, y, z \in X$ then $|y_k - z_k| \leq |x_k - y_k| + |x_k - z_k|$ for all k, and (as in Question 7) this gives $\frac{|y_k - z_k|}{1+|y_k - z_k|} \leq \frac{|x_k - y_k|}{1+|x_k - y_k|} + \frac{|x_k - z_k|}{1+|x_k - z_k|}$ for all k. Multiplying by $\frac{1}{2^k}$ and summing over k gives $d(y, z) \leq d(x, y) + d(x, z)$.