THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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# Tutorial 2

**1.** Let $A$ be a $4 \times 4$ matrix, and suppose that $v_1$, $v_2$, $v_3$ and $v_4$ are column vectors satisfying $Av_1 = 2v_1$, $Av_2 = 2v_2 + v_1$, $Av_3 = 3v_3$ and $Av_4 = 3v_4 + v_3$. Let $T$ be the matrix whose columns are $v_1$, $v_2$, $v_3$ and $v_4$ (in that order). Prove that

$$AT = T \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

**2.** For each of the following matrices $A$ find a nonsingular matrix $T$ such that $T^{-1}AT$ is diagonal.

(a) $A = \begin{pmatrix} 9 & -2 & 7 \\ 4 & -1 & 4 \\ -4 & 2 & -2 \end{pmatrix}$ (b) $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Check that it is possible in part (b) to choose $T$ in such a way that the sum of the squares of the entries in each column of $T$ is 1, and that if this is done then $T^{-1} = {}^{\mathrm{t}}T$.

**3.** Prove that if $A$ and $B$ are matrices such that $AB$ is defined then ${}^{\mathrm{t}}B\,{}^{\mathrm{t}}A$ is defined, and ${}^{\mathrm{t}}B\,{}^{\mathrm{t}}A = {}^{\mathrm{t}}(AB)$.

**4.** Let $A$ be a matrix satisfying ${}^{\mathrm{t}}A = A$ and let $u$ and $v$ be eigenvectors of $A$ with corresponding eigenvalues $\lambda$ and $\mu$. (That is, $u$ and $v$ are nonzero and $Au = \lambda u$ and $Av = \mu v$.) Prove that if $\lambda \neq \mu$ then $({}^{\mathrm{t}}u)v = 0$. (Hint: Show that $({}^{\mathrm{t}}u)A = \lambda({}^{\mathrm{t}}u)$, and then expand $({}^{\mathrm{t}}u)Av$ in two ways.)

Investigate the connection between this exercise and 2 (b).

**5.** Show that if $\alpha$ and $\beta$ are arbitrary complex numbers then $\overline{(\alpha+\beta)} = \overline{\alpha} + \overline{\beta}$ and $\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$, where the overline denotes complex conjugation (defined by $\overline{(x+iy)} = x - iy$ for all $x, y \in \mathbb{R}$, where $i = \sqrt{-1}$).

If $A$ is a complex matrix let $\overline{A}$ be the matrix whose entries are the complex conjugates of the entries of $A$. Use the previous part to show that $\overline{AB} = \overline{A}\,\overline{B}$ for all complex matrices $A$ and $B$ such that $AB$ exists.