THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 11

1. Let $\theta\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\theta\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Calculate the matrix of θ relative to the bases

$$\boldsymbol{d} = \left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}\right) \quad \text{and} \quad \boldsymbol{c} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

of $\mathbb{R}^3$ and $\mathbb{R}^2$.

Solution.

$$\theta\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} = 5\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and so the first column of the matrix of θ relative to these bases is $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$. Similarly

$$\theta\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = 11\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\theta\begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix} = -15\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and so the matrix of θ is

$$\begin{pmatrix} 5 & 11 & -15 \\ 1 & 3 & -1 \end{pmatrix}.$$

2. (i) Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $\phi\begin{pmatrix} x \\ y \end{pmatrix} = (\,1 \quad 2\,)\begin{pmatrix} x \\ y \end{pmatrix}$. Calculate the matrix of ϕ relative to the bases

$$\boldsymbol{c} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \quad \text{and} \quad \boldsymbol{b} = (-1)$$

of $\mathbb{R}^2$ and $\mathbb{R}$.

(ii) With ϕ as in (i) and θ as in Exercise 1 calculate $\phi\theta$ and its matrix relative the two given bases. Hence verify that $\mathrm{M}_{\boldsymbol{bd}}(\phi\theta) = \mathrm{M}_{\boldsymbol{bc}}(\phi)\,\mathrm{M}_{\boldsymbol{cd}}(\theta)$.

Solution.

(i) Since $\phi\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 = (-2)\times(-1)$ and $\phi\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (-3)\times(-1)$ the required matrix is $(\,-2 \quad -3\,)$.

(ii)

$$(\phi\theta)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\,1 \quad 2\,)\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\,5 \quad 1 \quad 9\,)\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Hence we find that

$$(\phi\theta)\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = (\,5 \quad 1 \quad 9\,)\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 13 = (-13)\times(-1),$$

and similarly

$$(\phi\theta)\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = (-31)\times(-1) \quad (\phi\theta)\begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix} = 33\times(-1)$$

so that the matrix of $\phi\theta$ relative to

$$\left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}\right) \qquad \text{and} \qquad (-1)$$

is $(\,-13 \quad -31 \quad 33\,)$. Now according to Theorem 7.5 this should equal the matrix of ϕ multiplied by the matrix of θ; that is, by our answers above,

$$(\,-2 \quad -3\,)\begin{pmatrix} 5 & 11 & -15 \\ 1 & 3 & -1 \end{pmatrix}.$$

Calculation shows that it is.

3. Suppose that $\theta\colon \mathbb{R}^6 \to \mathbb{R}^4$ is a linear transformation with kernel of dimension 2. Is θ surjective?

Solution.

(Dimension of image) = (dimension of domain) − (dimension of kernel); thus $\dim(\operatorname{im}\theta) = 6 - 2 = 4$. So the image is a 4-dimensional subspace of $\mathbb{R}^4$, hence equals the whole of $\mathbb{R}^4$. Hence θ is surjective.

4. For each of the following linear transformations calculate the dimensions of the kernel and image, and check that your answers are in agreement with the Main Theorem on Linear Transformations.

(*i*) $\theta\colon \mathbb{R}^4 \to \mathbb{R}^2$ given by $\theta\begin{pmatrix} x\\ y\\ z\\ w\end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 & 5\\ 1 & -1 & 1 & 0\end{pmatrix}\begin{pmatrix} x\\ y\\ z\\ w\end{pmatrix}$.

(*ii*) $\theta\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $\theta\begin{pmatrix} x\\ y\end{pmatrix} = \begin{pmatrix} 4 & -2\\ -2 & 1\\ -6 & 3\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}$.

(*iii*) $\theta\colon V \to V$ given by $\theta(p(x)) = p'(x)$, where V is the space of all polynomials over $\mathbb{R}$ of degree less than or equal to 3.

Solution.

(*i*). To find the kernel we must solve the equations

$$\begin{pmatrix} 2 & -1 & 3 & 5\\ 1 & -1 & 1 & 0\end{pmatrix}\begin{pmatrix} x\\ y\\ z\\ w\end{pmatrix} = \begin{pmatrix} 0\\ 0\end{pmatrix}. \tag{3}$$

Row reduction gives the reduced echelon matrix $\begin{pmatrix} 1 & 0 & 2 & 5\\ 0 & 1 & 1 & 5\end{pmatrix}$. Assigning the arbitrary values λ and μ to the free variables z and w gives the general solution

$$\begin{pmatrix} x\\ y\\ z\\ w\end{pmatrix} = \begin{pmatrix} -2\lambda-5\mu\\ -\lambda-5\mu\\ \lambda\\ \mu\end{pmatrix} = \lambda\begin{pmatrix} -2\\ -1\\ 1\\ 0\end{pmatrix} + \mu\begin{pmatrix} 5\\ 5\\ 0\\ 1\end{pmatrix} \tag{4}$$

so that the kernel is a 2-dimensional space. The image of θ is the column-space of the matrix and it can be seen that the first two columns (those corresponding to the non-free variables) form a basis for this columnspace. To see that they span, observe that since $\lambda = 1$ and $\mu = 0$ in (4) gives a solution to (3), we have $\begin{pmatrix} 2 & -1 & 3 & 5\\ 1 & -1 & 1 & 0\end{pmatrix}\begin{pmatrix} -2\\ -1\\ 1\\ 0\end{pmatrix} = \begin{pmatrix} 0\\ 0\end{pmatrix}$, which can alternatively be written as $\begin{pmatrix} 3\\ 1\end{pmatrix} = 2\begin{pmatrix} 2\\ 1\end{pmatrix} + \begin{pmatrix} -1\\ -1\end{pmatrix}$, showing that the third column is a linear combination of the first two. Similarly putting $\lambda = 0$ and $\mu = 1$ shows that the fourth column is a linear combination of the first two; so the third and fourth columns are in the space spanned by the first two. To see that the first two columns are linearly independent, observe that if we had $\alpha\begin{pmatrix} 2\\ 1\end{pmatrix} + \beta\begin{pmatrix} -1\\ -1\end{pmatrix} = \begin{pmatrix} 0\\ 0\end{pmatrix}$ then $\begin{pmatrix} \alpha\\ \beta\\ 0\\ 0\end{pmatrix}$ would be in the kernel of θ; however, from our description of the kernel above we see that the only element of the kernel which has zeros in the third and fourth entries is obtained by putting $\lambda = \mu = 0$, and this gives $\alpha = \beta = 0$. So the image has dimension 2, and $\dim(\operatorname{im}(\theta)) + \dim(\ker(\theta)) = 2+2 = 4 = \dim(\mathbb{R}^4)$, in agreement with the Main Theorem, since $\mathbb{R}^4$ is the domain of θ.

(*ii*). Use the same method as in the previous part. The row-reduced echelon matrix is $\begin{pmatrix} 1 & -1/2\\ 0 & 0\\ 0 & 0\end{pmatrix}$. Thus the kernel has dimension 1, $\begin{pmatrix} 2\\ 1\end{pmatrix}$ being a basis, and the image has dimension 1, the first column of the given matrix being a basis. This checks, since the domain of θ is $\mathbb{R}^2$.

(*iii*) $\dim V = 4$, since $(1, x, x^2, x^3)$ is a basis. The kernel of θ is the set of all polynomials in V with derivative zero; this is just the set of all constant polynomials, and is a one dimensional space. The image of θ is $\operatorname{im}\theta = \{\,\theta(a_0 + a_1x + a_2x^2 + a_3x^3) \mid a_i \in \mathbb{R}\,\} = \{\,a_1 + 2a_2x + 3a_3x^2 \mid a_i \in \mathbb{R}\,\}$ which is the set of all polynomials of degree less than or equal to 2, and has dimension 3. This checks, since $3 + 1 = 4$. Observe that we could also have used the same method as in parts (*i*) and (*ii*), using the fact that the matrix of θ relative to the above basis of V is $\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$.

5. Is it possible to find a 3×2 matrix A, a 2×2 matrix B and a 2×3 matrix C such that

$$ABC = \begin{pmatrix} 2 & 2 & 2\\ 3 & 3 & 0\\ 4 & 0 & 0\end{pmatrix}?$$

Solution.

No. The general theory shows that the rank of XY is always less than or equal to the rank of X and less than or equal to the rank of Y. So the $\operatorname{rank}(ABC) \le \operatorname{rank}(B)$ (for example). Since B is 2×2 its rank is at most 2. However, the given 3×3 matrix clearly has rank 3, since its rows are linearly independent.

6. Let V and W be finitely generated vector spaces of the same dimension and let $\theta\colon V \to W$ be a linear transformation. Use the Main Theorem on Linear Transformations to prove that θ is injective if and only if it is surjective.

Solution.

Let $n = \dim V = \dim W$. The Main Theorem gives

$$\dim\ker\theta + \dim\operatorname{im}\theta = n,$$

and so we conclude that $\dim\ker\theta = 0$ if and only if $\dim\operatorname{im}\theta = n$.

A linear transformation is injective if and only if its kernel is 0 (Proposition 3.15), and $\{0\}$ is the one and only subspace of V of dimension zero (see 4.1.5). So θ is injective if and only if $\dim\ker\theta = 0$.

The one and only n dimensional subspace of an n dimensional space is the space itself (see 4.11), and so $\dim\operatorname{im}\theta = n$ if and only if $\operatorname{im}\theta = W$. Since by definition θ is surjective if and only if $\operatorname{im}\theta = W$, we conclude that θ is surjective if and only if $\dim\operatorname{im}\theta = n$.

Combining the conclusions of these three paragraphs completes the proof.