THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 10

1. Prove that isomorphic vector spaces have the same dimension.
(Hint: Use Theorem 4.17. This was proved in Exercise 5 of Tutorial 4.)

Solution.

Let V and W be isomorphic vector spaces and let $\theta\colon V \to W$ be an isomorphism. That is, θ is a bijective linear transformation. Let $v_1, v_2, \ldots, v_n$ be a basis for V. By 4.17 (ii) the elements $\theta(v_1), \theta(v_2), \ldots, \theta(v_n)$ span W (since θ is surjective), and by 4.17 (i) they are linearly independent (since θ is injective). So these elements form a basis for W, and we see that bases of W have the same number of elements as do bases of V.

2. Is it possible to find subspaces U, V and W of $\mathbb{R}^4$ such that

$$\mathbb{R}^4 = U \oplus V = V \oplus W = W \oplus U\,?$$

Solution.

Yes; for instance, define U, V and W to be (respectively)

$$\left\{ \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}, \quad \left\{ \begin{pmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}, \quad \left\{ \begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}.$$

Each of these is a subspace of dimension two: it can be seen that

$$\boldsymbol{b} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right), \quad \boldsymbol{c} = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right), \quad \boldsymbol{d} = \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

are bases of U, V and W respectively. Now since $U \cap V = \{0\}$ the sum $U + V$ is direct, and its dimension is therefore equal to $\dim U + \dim V = 4$. The only 4-dimensional subspace of $\mathbb{R}^4$ is $\mathbb{R}^4$ itself; so we conclude that $U \oplus V = \mathbb{R}^4$. (Indeed, combining the bases $\boldsymbol{b}$ of U and $\boldsymbol{c}$ of V gives the standard basis of $\mathbb{R}^4$.) Since it is also true that $U \cap W = \{0\}$ and $V \cap W = \{0\}$ it follows that $U \oplus W = V \oplus W = \mathbb{R}^4$ as well.

3. (*i*) Let V and W be vector spaces over F. Show that the Cartesian product of V and W (see §1b) becomes a vector space if addition and scalar multiplication are defined in the natural way. (This space is called the *external direct sum* of V and W, and is sometimes denoted by '$V \dotplus W$'.)

(*ii*) Show that $V' = \{\,(v,0) \mid v \in V\,\}$ and $W' = \{\,(0,w) \mid w \in W\,\}$ are subspaces of $V \dotplus W$ with $V' \cong V$ and $W' \cong W$, and that $V \dotplus W = V' \oplus W'$.

(*iii*) Prove that $\dim(V \dotplus W) = \dim V + \dim W$.

Solution.

(*i*) Elements of $V \dotplus W$ are ordered pairs (v,w) with $v \in V$ and $w \in W$. Addition and scalar multiplication are defined by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \qquad \lambda(v_1, w_1) = (\lambda v_1, \lambda w_1)$$

for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$ and all $\lambda \in F$. To prove that this gives a vector space is simply a matter of checking the axioms. The zero element of $V \dotplus W$ is the ordered pair $(0,0)$ (where the first 0 is the zero of V and the second the zero of W). The negative of (v,w) is $(-v,-w)$.

For all $\lambda, \mu \in F$ and all $v \in V$ and $w \in W$ we have

$$\begin{aligned} (\lambda+\mu)(v,w) &= ((\lambda+\mu)v, (\lambda+\mu)w) && \text{(definition of scalar multiplication)} \\ &= (\lambda v + \mu v, \lambda w + \mu w) && \text{(vector space axioms in } V, W) \\ &= (\lambda v, \lambda w) + (\mu v, \mu w) && \text{(definition of addition)} \\ &= \lambda(v,w) + \mu(v,w) && \text{(definition of scalar multiplication)} \end{aligned}$$

proving Axiom (vii) of Definition 2.3. The other axioms can be done similarly, in each case making use of the fact that the axiom in question is satisfied in V and in W (since it is given that V and W are vector spaces).

(*ii*) Define $\theta\colon V \to V \dotplus W$ by $\theta(v) = (v,0)$ for all $v \in V$. Then for all $u, v \in V$ and $\lambda, \mu \in F$ we have

$$\theta(\lambda u + \mu v) = (\lambda u + \mu v, 0) = \lambda(u,0) + \mu(v,0) = \lambda\theta(u) + \mu\theta(v).$$

Hence θ is a linear transformation. The kernel of θ consists of all $v \in V$ such that $(v,0)$ is the zero element of $V \dotplus W$. Hence $\ker\theta = \{0\}$, and it follows that θ is injective. The image of θ is the subset of $V \dotplus W$ consisting of all elements of the form $\theta(v)$ for $v \in V$; thus $\operatorname{im}\theta = V'$. By 3.14 we deduce that V' is a subspace of $V \dotplus W$.

Define $\theta'\colon V \to V'$ by $\theta'(v) = \theta(v)$ for all v. That is, θ' is just θ with its codomain cut down to coincide with its image. This makes θ' surjective, and it is also injective (since θ is). Hence θ' is an isomorphism, and $V' \cong V$.

Virtually identical arguments using the map $w \mapsto (0, w)$ show that W' is a subspace and isomorphic to W. Since an arbitrary element of $V \dotplus W$ has the form $(v, w) = (v, 0) + (0, w) \in V' + W'$ we see that $V \dotplus W = V' + W'$, and since $(v, 0) = (0, w)$ implies $v = w = 0$ we see that $V' \cap W' = \{0\}$. Hence $V \dotplus W = V' \oplus W'$.

(*iii*) Since $V' \cong V$ and $W' \cong W$ we deduce that $\dim V' = \dim V$ and $\dim W' = \dim W$ (by Exercise 1). But since $V \dotplus W = V' \oplus W'$ Theorem 6.9 gives $\dim(V \dotplus W) = \dim V' + \dim W'$, whence the result.

4. Let S and T be subspaces of a vector space V and let U be a subspace of T such that $T = (S \cap T) \oplus U$. Prove that $S + T = S \oplus U$ (see Tutorial 3 for the definition of $S + T$), and hence deduce that

$$\dim(S + T) = \dim S + \dim T - \dim(S \cap T).$$

Solution.

From an earlier tutorial we know that $S + T$ is a subspace of V. If $s \in S$ then $s = s + 0 \in S + T$; so $S \subseteq S + T$. Similarly $T \subseteq S + T$, and since $U \subseteq T$ we have $U \subseteq S + T$. So S and U are subspaces of $S + T$, and we must show that $S + U = S + T$ and $S \cap U = \{0\}$.

Let $x \in S + T$. Then $x = s + t$ for some $s \in S$, $t \in T$. Since $T = (S \cap T) \oplus U$ there exist $r \in S \cap T$, $u \in U$ with $t = r + u$. Since $r \in S \cap T \subseteq S$ and $s \in S$ we have $s + r \in S$, and therefore

$$x = s + (r + u) = (s + r) + u \in S + U.$$

Since x was arbitrary we have shown that all elements of $S + T$ lie in the subspace $S + U$ of $S + T$; thus $S + U = S + T$.

Let $a \in S \cap U$. Then $a \in S$ and $a \in U \subseteq T$; so $a \in S \cap T$. But $a \in U$; so $a \in (S \cap T) \cap U$. Because the sum of $S \cap T$ and U is direct we have that $(S \cap T) \cap U = \{0\}$, and therefore $a = 0$. But a was an arbitrary element of $S \cap U$, and so we have shown that $S \cap U = \{0\}$, as required.

Alternatively, making use of some easily proved facts about adding subspaces, we have

$$S + T = S + ((S \cap T) + U) = (S + (S \cap T)) + U = S + U$$

(where $S + (S \cap T) = S$ holds since $S \cap T \subseteq S$) and

$$S \cap U = S \cap (T \cap U) = (S \cap T) \cap U = \{0\}$$

(where $U = T \cap U$ holds since $U \subseteq T$.)

Since $T = (S \cap T) \oplus U$ we have

(1) $$\dim T = \dim(S \cap T) + \dim U.$$

Since $S + T = S \oplus U$ we have

(2) $$\dim(S + T) = \dim S + \dim U.$$

Eliminating $\dim U$ from equations (1) and (2) gives

$$\dim(S + T) = \dim S + \dim T - \dim(S \cap T).$$

5. (*i*) Let S and T be subspaces of a vector space V. Prove that $(s, t) \mapsto s + t$ defines a linear transformation from $S \dotplus T$ to V which has image $S + T$ and kernel isomorphic to $S \cap T$.

(*ii*) The Main Theorem on Linear Transformations (see p. 158 of the book) asserts that if V is a finitely generated vector space and θ a linear transformation from V to another space W, then the sum of the dimensions of $\ker \theta$ and $\operatorname{im} \theta$ equals the dimension of V. Use this and Part (*i*) to give another proof that $\dim(S + T) + \dim(S \cap T) = \dim S + \dim T$.

Solution.

Since every element of $S \dotplus T$ is uniquely expressible in the form (s, t) with $s \in S$ and $t \in T$, and since S and T are subspaces of the vector space V, the formula $\theta(s, t) = s + t$ defines a function from $S \dotplus T$ to V. Now if $(s, t), (s', t') \in S \dotplus T$ and λ is a scalar then

$$\begin{aligned} \theta((s, t) + (s', t')) &= \theta(s + s', t + t') = (s + s') + (t + t') \\ &= (s + t) + (s' + t') = \theta(s, t) + \theta(s', t') \end{aligned}$$

(by definition of θ, definition of addition in $S \dotplus T$ and properties of addition in the vector space V), and

$$\theta(\lambda(s, t)) = \theta(\lambda s, \lambda t) = \lambda s + \lambda t = \lambda(s + t) = \lambda\theta(s, t)$$

similarly. Hence θ is linear.

The image of θ is the set of all elements of V of the form $\theta(s, t) = s + t$ with $s \in S$ and $t \in T$; that is, $\operatorname{im} \theta = S + T$. The kernel of θ consists of all (s, t) such that $s \in S$, $t \in T$ and $s + t = 0$. For these conditions to be satisfied we must have $s = -t \in T$, and hence $s \in S \cap T$. Conversely, if $x \in S \cap T$ then $(x, -x)$ is in the kernel. So $\ker \theta = \{\, (x, -x) \mid x \in S \cap T \,\}$. Hence the mapping $\phi \colon S \cap T \to \ker \theta$ defined by $\phi(x) = (x, -x)$ is surjective. It is also injective, since $(x, -x) = (y, -y)$ implies $x = y$. Finally, ϕ is linear since

$$\begin{aligned} \phi(\lambda x + \mu y) &= (\lambda x + \mu y, -(\lambda x + \mu y)) = (\lambda x, -\lambda x) + (\mu y, -\mu y) \\ &= \lambda(x, -x) + \mu(y, -y) = \lambda\phi(x) + \mu\phi(y) \end{aligned}$$

for all x, $y \in S \cap T$ and all scalars λ and μ. Hence $\ker \theta \cong S \cap T$.

By the Main Theorem, $\dim \ker \theta + \dim \operatorname{im} \theta = \dim(S \dotplus T)$. Since $\ker \theta \cong S \cap T$ we know (by Exercise 1) that $\dim \ker \theta = \dim(S \cap T)$, and by Exercise 3 we know that $\dim S \dotplus T = \dim S + \dim T$. Combining all this with $\operatorname{im} \theta = S + T$ gives $\dim(S \cap T) + \dim(S + T) = \dim S + \dim T$, as required.