THE UNIVERSITY OF SYDNEY

MATH2008 Introduction to Modern Algebra

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2008/)

Semester2, 2003 — Lecturer: R. Howlett

Tutorial 1

Please be sure to bring your lecture notes to the tutorial.

At the end of the semester you will be awarded a mark for tutorial participation. If you merely attend the tutorial without working on the problems then you will not receive credit for that tutorial.

The system log files record your attendance at the computer tutorials. These will also be consulted when determining your tutorial mark.

1. Let $\underset{\sim}{u} = (1, 0, -5, 7)^T$ and $\underset{\sim}{v} = (21, 2, 2, -2)^T$. Find the lengths of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ and the angle between them.

Solution.

$\|\underset{\sim}{u}\|^2 = \underset{\sim}{u} \cdot \underset{\sim}{u} = 1^2 + 0^2 + (-5)^2 + 7^2 = 75$; so $\|\underset{\sim}{u}\| = 5\sqrt{3} \approx 8.66$. Simlarly, $\|\underset{\sim}{v}\| = \sqrt{453} \approx 21.28$, and $\underset{\sim}{u} \cdot \underset{\sim}{v} = 1 \times 21 - 5 \times 2 - 7 \times 2 = -3$. The angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is $\arccos(\frac{-3}{5\sqrt{3}\sqrt{453}}) \approx 1.59$ radians (which is about $90^\circ\ 56'$).

2. The four points $(0,0,0)$, $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ are the vertices of a tetrahedron in $\mathbb{R}^3$.

(*i*) Show that all six edges of this tetrahedron have the same length.

(*ii*) Given that the centre of this tetrahedron is $(1/2, 1/2, 1/2)$, calculate the angle between two rays joining the centre to two of the vertices. Check that you get the same answer whichever two vertices you choose.

(This tetrahedron can be seen as a model of a methane molecule, with a carbon atom at the centre and hydrogen atoms at the vertices. The angle in Part (*ii*) is the "bond angle".)

Solution.

(*i*) Let $\underset{\sim}{v}_1 = (1,1,0)^T$, $\underset{\sim}{v}_2 = (1,0,1)^T$, $\underset{\sim}{v}_3 = (0,1,1)^T$, the vectors representing three of the vertices; the remaining vertex is given by the vector $\underset{\sim}{v}_0 = \underset{\sim}{0}$. Now $d(\underset{\sim}{v}_1, \underset{\sim}{v}_0) = \|\underset{\sim}{v}_1\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$, and by a similar calculation $d(\underset{\sim}{v}_2, \underset{\sim}{v}_0) = d(\underset{\sim}{v}_3, \underset{\sim}{v}_0) = \sqrt{2}$. We also have $d(\underset{\sim}{v}_1, \underset{\sim}{v}_2) = \|\underset{\sim}{v}_1 - \underset{\sim}{v}_2\| = \|(0,1,-1)\| = \sqrt{2}$, and again similar calculations give $d(\underset{\sim}{v}_1, \underset{\sim}{v}_3) = \|(1,0,-1)\| = \sqrt{2}$ and $d(\underset{\sim}{v}_2, \underset{\sim}{v}_3) = \|(1,-1,0)\| = \sqrt{2}$.

(*ii*) Let $\underset{\sim}{w} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$. Then $d(\underset{\sim}{v}_0, \underset{\sim}{w}) = \|\underset{\sim}{w}\| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{3}}{2}$. Moreover, $d(\underset{\sim}{v}_1, \underset{\sim}{w}) = \|\underset{\sim}{v}_1 - \underset{\sim}{w}\| = \|(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\| = \frac{\sqrt{3}}{2}$, and we find similarly that $d(\underset{\sim}{v}_2, \underset{\sim}{w}) = d(\underset{\sim}{v}_3, \underset{\sim}{w}) = \frac{\sqrt{3}}{2}$. The rays from the centre to the vertices $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ correspond to the vectors $\underset{\sim}{v}_1 - \underset{\sim}{w} = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ and $\underset{\sim}{v}_2 - \underset{\sim}{w} = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. The dot product of these two is $\frac{1}{4} - \frac{1}{4} - \frac{1}{4}$, and so the cosine of the angle between them is $(-\frac{1}{4}/\frac{3}{4})$ (since they each have length $\frac{\sqrt{3}}{2}$). So the angle is $\arccos(-1/3) \approx 1.91$ radians, or $109^\circ\ 28'$. The other pairs of rays give the same angle.

3. Prove the following properties of the dot product.

(*i*) $(\underset{\sim}{u} + \underset{\sim}{v}) \cdot \underset{\sim}{w} = \underset{\sim}{u} \cdot \underset{\sim}{w} + \underset{\sim}{v} \cdot \underset{\sim}{w}$ for all $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathbb{R}^n$.

(*ii*) $\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{u}$ for all $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^n$.

(*iii*) $k(\underset{\sim}{u} \cdot \underset{\sim}{v}) = (k\underset{\sim}{u}) \cdot \underset{\sim}{v} = \underset{\sim}{u} \cdot (k\underset{\sim}{v})$ for all $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^n$ and $k \in \mathbb{R}$.

(*iv*) $\underset{\sim}{u} \cdot \underset{\sim}{u} \geq 0$, and if $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$ then $\underset{\sim}{u} = \underset{\sim}{0}$, for all $\underset{\sim}{u} \in \mathbb{R}^n$.

Solution.

Let $\underset{\sim}{u} = (u_1, u_2, \ldots, u_n)^T$, $\underset{\sim}{v} = (v_1, v_2, \ldots, v_n)^T$ $\underset{\sim}{w} = (w_1, w_2, \ldots, w_n)^T$. Then $\underset{\sim}{u} + \underset{\sim}{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)^T$, and

$$\begin{aligned}(\underset{\sim}{u} + \underset{\sim}{v}) \cdot \underset{\sim}{w} &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + \cdots + (u_n + v_n)w_n \\ &= (u_1w_1 + v_1w_1) + (u_2w_2 + v_2w_2) + \cdots + (u_nw_n + v_nw_n) \\ &= (u_1w_1 + u_2w_2 + \cdots + u_nw_n) + (v_1w_1 + v_2w_2 + \cdots + v_nw_n) \\ &= \underset{\sim}{u} \cdot \underset{\sim}{w} + \underset{\sim}{v} \cdot \underset{\sim}{w}\end{aligned}$$

proving (*i*). Similarly, $\underset{\sim}{u} \cdot \underset{\sim}{v} = \sum_{i=1}^n u_iv_i = \sum_{i=1}^n v_iu_i = \underset{\sim}{v} \cdot \underset{\sim}{u}$, proving (*ii*). And $k(\underset{\sim}{u} \cdot \underset{\sim}{v}) = k\sum_{i=1}^n u_iv_i = \sum_{i=1}^n (ku_i)v_i = (k\underset{\sim}{u}) \cdot \underset{\sim}{v} = \sum_{i=1}^n u_i(kv_i) = \underset{\sim}{u} \cdot (k\underset{\sim}{v})$, proving (*iii*). As for (*iv*), $\underset{\sim}{u} \cdot \underset{\sim}{u} = \sum_{i=1}^n u_i^2 \geq 0$, since all u_i^2 are nonnegative, and if any u_i^2 is strictly positive then $\underset{\sim}{u} \cdot \underset{\sim}{u} > 0$. So if $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$ then each u_i^2 must be zero, giving $\underset{\sim}{v} = \underset{\sim}{0}$.

4. If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are points in $\mathbb{R}^n$ then the *distance* $d(\underset{\sim}{u}, \underset{\sim}{v})$ from $\underset{\sim}{u}$ to $\underset{\sim}{v}$ is defined by $d(\underset{\sim}{u}, \underset{\sim}{v}) = \|\underset{\sim}{u} - \underset{\sim}{v}\|$. The *triangle inequality* says that $d(\underset{\sim}{u}, \underset{\sim}{v}) + d(\underset{\sim}{v}, \underset{\sim}{w}) \geq d(\underset{\sim}{u}, \underset{\sim}{w})$ for all $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathbb{R}^n$. Fill in the details of the following proof of this fact.

(*i*) Use properties of dot products to show that $\|\underset{\sim}{x} + \underset{\sim}{y}\|^2 = \|\underset{\sim}{x}\|^2 + 2\underset{\sim}{x} \cdot \underset{\sim}{y} + \|\underset{\sim}{y}\|^2$, and deduce that $\|\underset{\sim}{x} + \underset{\sim}{y}\|^2 \leq \|\underset{\sim}{x}\|^2 + 2\|\underset{\sim}{x}\|\|\underset{\sim}{y}\| + \|\underset{\sim}{y}\|^2$. (Use the Cauchy-Schwarz inequality for this step.)

(*ii*) Conclude that $\|\underset{\sim}{x} + \underset{\sim}{y}\| \leq \|\underset{\sim}{x}\| + \|\underset{\sim}{y}\|$. (This is also a fact that you should remember.)

(*iii*) In (*ii*) replace $\underset{\sim}{x}$ by $\underset{\sim}{u} - \underset{\sim}{v}$ and $\underset{\sim}{y}$ by $\underset{\sim}{v} - \underset{\sim}{w}$.

Solution.

The Cauchy-Schwarz inequality says that $|\utilde{x} \cdot \utilde{y}| \le \|\utilde{x}\|\,\|\utilde{y}\|$ for all $\utilde{x}$, $\utilde{y} \in \mathbb{R}^n$. Since obviously $\utilde{x} \cdot \utilde{y} \le |\utilde{x} \cdot \utilde{y}|$, it follows that for all $\utilde{x}$, $\utilde{y} \in \mathbb{R}^n$,

$$\begin{aligned}
\|\utilde{x} + \utilde{y}\|^2 &= (\utilde{x} + \utilde{y}) \cdot (\utilde{x} + \utilde{y}) \\
&= \utilde{x} \cdot \utilde{y} + 2\utilde{x} \cdot \utilde{y} = \utilde{y} \cdot \utilde{y} \\
&= \|\utilde{x}\|^2 + 2\utilde{x} \cdot \utilde{y} + \|\utilde{y}\|^2 \\
&\le \|\utilde{x}\|^2 + 2\|\utilde{x}\|\,\|\utilde{y}\| + \|\utilde{y}\|^2 \\
&= (\|\utilde{x}\| + \|\utilde{y}\|)^2.
\end{aligned}$$

Since $\|\utilde{x} + \utilde{y}\|$ and $\|\utilde{x}\| + \|\utilde{y}\|$ are both nonnegative, taking square roots gives $\|\utilde{x} + \utilde{y}\| \le \|\utilde{x}\| + \|\utilde{y}\|$, as required for Part (*ii*). Since this holds for all $\utilde{x}$ and $\utilde{y} \in \mathbb{R}^n$, it holds when $\utilde{x} = \utilde{u} - \utilde{v}$ and $\utilde{y} = \utilde{v} - \utilde{w}$, for any $\utilde{u}$, $\utilde{v}$, $\utilde{w} \in \mathbb{R}^n$. Now $\utilde{x} + \utilde{y} = (\utilde{u} - \utilde{v}) + (\utilde{v} - \utilde{w}) = \utilde{u} - \utilde{w}$; so $\|\utilde{u} - \utilde{w}\| \le \|\utilde{u} - \utilde{v}\| + \|\utilde{v} - \utilde{w}\|$, as required.

5. Let B be a square matrix.

(*i*) Show that $B + B^T$ is symmetric.

(*ii*) Show that $B - B^T$ is skew-symmetric. (A matrix X is called *skew-symmetric* if $X^T = -X$.)

Solution.

We use the properties of the transpose operation that were stated in lectures. In particular, $(X + Y)^T = X^T + Y^T$ whenever X and Y are matrices of the same shape. Here since B is square we see that B^T has the same shape as B, and so $B + B^T$ and $B - B^T$ both make sense.

(*i*) $(B + B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T$.

(*ii*) $(B - B^T)^T = B^T - (B^T)^T = B^T - B = -(B - B^T)$.

6. Let $\utilde{v}$ be a fixed non-zero vector in $\mathbb{R}^n$. Let W be the set of all vectors in $\mathbb{R}^n$ orthogonal to $\utilde{v}$. Show that W is a subspace of $\mathbb{R}^n$.

Solution.

Recall that a subset of $\mathbb{R}^n$ is a subspace if and only if it contains the zero vector and is closed under addition and scalar multiplication. So to prove that W is a subspace we must prove the following facts:

(a) $\utilde{0} \in W$;

(b) if $\utilde{x}$, $\utilde{y} \in W$ then $\utilde{x} + \utilde{y} \in W$;

(c) if $\utilde{x} \in W$ then $k\utilde{x} \in W$ for all scalars k.

We have $\utilde{0} \cdot \utilde{v} = 0$, and therefore $\utilde{0} \in W$. So (a) holds. Let $\utilde{x}$, $\utilde{y} \in W$. Then $\utilde{x} \cdot \utilde{v} = 0$ and $\utilde{y} \cdot \utilde{v} = 0$, and we see that $(\utilde{x} + \utilde{y}) \cdot \utilde{v} = \utilde{x} \cdot \utilde{v} + \utilde{y} \cdot \utilde{v} = 0 + 0 = 0$.

It follows that $\utilde{x} + \utilde{y} \in W$. This is true for all $\utilde{x}$, $\utilde{y} \in W$; so (b) holds. Finally, suppose that $\utilde{x} \in W$ and that k is any scalar. Then $\utilde{x} \cdot \utilde{v} = 0$, and $(k\utilde{x}) \cdot \utilde{v} = k(\utilde{x} \cdot \utilde{v}) = k0 = 0$. Hence $k\utilde{x} \in W$. Thus (c) holds too, and so W is a subspace.

7. Let $\utilde{u}$, $\utilde{v} \in \mathbb{R}^n$ with $\utilde{u} \ne \utilde{0}$. Show that the Cauchy-Schwarz inequality becomes an equality (that is, $|\utilde{u} \cdot \utilde{v}| = \|\utilde{u}\|\,\|\utilde{v}\|$) only if $\utilde{v} = \lambda\utilde{u}$ for some scalar λ. (Hint: Calculate $(\lambda\utilde{u}+\utilde{v}) \cdot (\lambda\utilde{u}+\utilde{v})$ when $\lambda = -(\utilde{u} \cdot \utilde{v})/(\utilde{u} \cdot \utilde{u})$.) Check also the converse statement: if $\utilde{v} = \lambda\utilde{u}$ for some scalar λ then $|\utilde{u} \cdot \utilde{v}| = \|\utilde{u}\|\,\|\utilde{v}\|$.

Solution.

Suppose that $|\utilde{u} \cdot \utilde{v}| = \|\utilde{u}\|\,\|\utilde{v}\|$ and $\utilde{u} \ne \utilde{0}$. If λ is any scalar then

$$(\lambda\utilde{u} + \utilde{v}) \cdot (\lambda\utilde{u} + \utilde{v}) = \lambda^2\utilde{u} \cdot \utilde{u} + 2\lambda\utilde{u} \cdot \utilde{v} + \utilde{v} \cdot \utilde{v},$$

and if we put $\lambda = -(\utilde{u} \cdot \utilde{v})/(\utilde{u} \cdot \utilde{u})$ then we find that

$$\begin{aligned}
(\lambda\utilde{u} + \utilde{v}) \cdot (\lambda\utilde{u} + \utilde{v}) &= \left(\frac{\utilde{u} \cdot \utilde{v}}{\utilde{u} \cdot \utilde{u}}\right)^2 \utilde{u} \cdot \utilde{u} - 2\left(\frac{\utilde{u} \cdot \utilde{v}}{\utilde{u} \cdot \utilde{u}}\right)\utilde{u} \cdot \utilde{v} + \utilde{v} \cdot \utilde{v} \\
&= \frac{(\utilde{u} \cdot \utilde{v})^2}{\utilde{u} \cdot \utilde{u}} - 2\frac{(\utilde{u} \cdot \utilde{v})^2}{\utilde{u} \cdot \utilde{u}} + \utilde{v} \cdot \utilde{v} \\
&= -\frac{(\utilde{u} \cdot \utilde{v})^2}{\utilde{u} \cdot \utilde{u}} + \utilde{v} \cdot \utilde{v} \\
&= -\frac{(\utilde{u} \cdot \utilde{v})^2}{\|\utilde{u}\|^2} + \|\utilde{v}\|^2.
\end{aligned}$$

But since we are assuming that $|\utilde{u} \cdot \utilde{v}| = \|\utilde{u}\|\,\|\utilde{v}\|$, this last expression equals $-\frac{\|\utilde{u}\|^2\|\utilde{v}\|^2}{\|\utilde{u}\|^2} + \|\utilde{v}\|^2 = 0$. Thus $(\lambda\utilde{u} + \utilde{v}) \cdot (\lambda\utilde{u} + \utilde{v}) = 0$, which means that $\lambda\utilde{u} + \utilde{v} = \utilde{0}$, and so $v = -\lambda u$. (The hint should really have told you to look at $(\lambda\utilde{u} - \utilde{v}) \cdot (\lambda\utilde{u} - \utilde{v})$ with $\lambda = (\utilde{u} \cdot \utilde{v})/(\utilde{u} \cdot \utilde{u})$. Then we would have obtained $v = \lambda u$ rather than $v = -\lambda u$.)

Conversely, if $v = \lambda u$ then

$$|\utilde{u} \cdot \utilde{v}| = |\lambda(\utilde{u} \cdot \utilde{u})| = |\lambda|\,\|\utilde{u}\|^2 = \|\utilde{u}\|\,\|\lambda\utilde{u}\| = \|\utilde{u}\|\,\|\utilde{v}\|,$$

as required.