THE UNIVERSITY OF SYDNEY

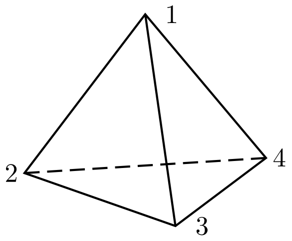
MATH2008 Introduction to Modern Algebra

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2008/)

Semester 2, 2003 Lecturer: R. Howlett

Computer Tutorial 10

1. This question is about the group of symmetries of the tetrahedron with vertices labelled 1, 2, 3 and 4 as shown below. Use MAGMA to set up the group $G := \mathrm{Sym}(4)$ of all permutations of $\{1, 2, 3, 4\}$.



Every rotational symmetry of the tetrahedron corresponds to a permutation of $\{1, 2, 3, 4\}$.

(*i*) Find a (nontrivial) rotational symmetry that fixes the vertex 1 and another that fixes the vertex 2, and find the corresponding permutations.

(*ii*) Let H be the subgroup of G generated by the permutations you found in Part (*i*). Get MAGMA to print out all the elements of H, and show that the order of H is 12.

(*iii*) Describe each element of H geometrically (e.g. as a rotation about an axis).

(*iv*) List the order of each element of H.

(*v*) Find a subgroup of H of order 4.

Solution.

Let ℓ_1 be the line through vertex 1 and the central point of the face 234. The rotations about the axis ℓ_1 through 120° and 240° are symmetries of the tetrahedron fixing vertex 1. The corresponding permutations are $(2, 3, 4)$ and $(2, 4, 3)$. Similarly, if ℓ_2 is the line through vertex 2 and the centroid of the face 134 then rotations about ℓ_2 through 120° and 240° are symmetries of the tetrahedron fixing vertex 2. The corresponding permutations are $(1, 3, 4)$ and $(1, 4, 3)$. For Part (*i*) of the question I chose $(2, 3, 4)$ and $(1, 3, 4)$. There are three other possible choices that would be equally valid.

```
> G:=Sym(4);
> x:=G!(1,3,4);
> y:=G!(2,3,4);
> H:=sub< G | x,y >;
> H;
Permutation group H acting on a set of cardinality 4
    (1, 3, 4)
    (2, 3, 4)
> Set(H);
{
    (1, 2)(3, 4),
    (1, 3, 2),
    (1, 3)(2, 4),
    (1, 2, 4),
    (1, 4, 3),
    (1, 3, 4),
    (1, 4, 2),
    Id(H),
    (1, 4)(2, 3),
    (2, 4, 3),
    (1, 2, 3),
    (2, 3, 4)
}
> for t in H do
for> "the order of",t,"is",Order(t);
for> end for;
the order of Id(H) is 1
the order of (1, 3, 4) is 3
the order of (1, 4, 3) is 3
the order of (1, 2, 3) is 3
the order of (2, 3, 4) is 3
the order of (1, 3)(2, 4) is 2
the order of (1, 4, 2) is 3
the order of (1, 2)(3, 4) is 2
the order of (2, 4, 3) is 3
the order of (1, 3, 2) is 3
the order of (1, 4)(2, 3) is 2
the order of (1, 2, 4) is 3
```

Sure enough, H has 12 elements. Four of them have been described above. The permutations $(1, 2, 4)$ and $(1, 4, 2)$ correspond to rotations through 120° and 240° about ℓ_3, the line joining vertex 3 to the centroid of 124. Similarly, $(1, 2, 3)$ and $(1, 3, 2)$ correspond to rotations through 120° and 240° about ℓ_4, the line joining vertex 4 to the centroid of 123. The identity is a rotation through 0° (about any axis). The remaining three elements of H are all half-turns: rotations through 180°. For the permutation $(1, 2)(3, 4)$ the axis is the line joining the mid-point of 12 to the midpoint of 34. Similarly, for $(1, 3)(2, 4)$ the axis is the line joining the mid-point of 13 to the midpoint of 24, and for $(1, 4)(2, 3)$ the axis is the line joining the mid-point of 14 to the midpoint of 23.

By Sylow's Theorem H must have a subgroup of order 4, since 4 is the largest power of the prime 2 that is a divisor of 12, the order of H. An element of order k generates a cyclic subgroup of order k, and by Lagrange's Theorem the order of a subgroup has to be a divisor of the order of the group. So the order of any element of a group of order 4 must be a divisor of 4. Now in H there are only four elements whose orders are divisors of 4: the three elements of order 2 and the identity (of order 1). So these four elements are the only ones that can possibly be contained in a group of order 4. But H does have a subgroup of order 4, which certainly contains four elements of H. So it must be these four. So

$$\{\mathrm{id}, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

is a subgroup of H of order 4.

2. In MAGMA, define `G` to be the group Sym(4), and define `p1:={{1,4},{2,3}};`.

(*i*) How many elements does the set `p1` have? Check your answer with MAGMA. (Use `#p1;`)

(*ii*) Define `P := p1^G;`, and then get MAGMA to print `P`. (Here `p1^G` means the set of everything that `p1` can be changed into by applying a permutation of $\{1,2,3,4\}$. This same example will be discussed in Q1 of Tutorial 10.

(*iii*) How many elements does `P` have? Check your answer with MAGMA (via the command `#P;`).

(*iv*) Each element of `P` corresponds to a partitioning of the set $\{1,2,3,4\}$ into two subsets of size 2. (Each such partitioning corresponds to a way of pairing up four tennis players for a game of doubles. Thus `p1` above corresponds to players 1 and 4 teaming up against players 2 and 3.) Define now `p2 :={{2,4},{1,3}};` and `p3 :={{3,4},{1,2}};`, so that `P` is `{p1, p2, p3}`. Observe that `p1` is a set with two elements, both of which are themselves sets. And `P` is a set whose elements are sets whose elements are sets.

(*v*) Put `x:=G!(1,4,3,2);`, and get MAGMA to print `p1^x`, `p2^x` and `p3^x`. Hence find the permutation of `{p1, p2, p3}` derived from the permutation `x` of $\{1,2,3,4\}$.

(*vi*) Each permutation of $\{1,2,3,4\}$ gives rise to a permutation of `{p1, p2, p3}`; so we have a function f from the group of all permutations of $\{1,2,3,4\}$ to the group of all permutations of `{p1, p2, p3}`. This function is, in fact, a homomorphism. The MAGMA command
`f,L,K := Action(G,P);`
defines `f` to be this homomorphism, `L` to be the image of `f`, and `K` to be the kernel of `f`. After typing this command, get MAGMA to print `f`, `L` and `K`.

(*vii*) Type the MAGMA command `f(x);`. The response should agree with your answer to Part (v).

(*viii*) Find the permutations of `{p1, p2, p3}` corresponding to each of the permutations $(1,4)$, $(1,3,2)$, $(1,2,3,4)$, $(1,3)$, $(2,4,3)$, by using commands such as `f(G!(1,4))`.

(*ix*) Find the permutations of `{p1,p2,p3}` corresponding to each of the permutations $(1,2)(3,4)$, $(1,3)(2,4))$ and $(1,3)(4,2)$. Note that these three permutations are all in the group `K`. Print `Set(K)` to confirm this.

(*x*) Put `A:= { x*k : k in K }`, and then do the following loop:

```
for t in A do
f(t);
end for;
```

What do you notice about the answer? Put `B := { G!(1,4)*k : k in K }`, and do a similar for loop. Observe that you again get the same answer four times. Do some more similar loops.

Solution.

`#p1` is 2. The two elements of `p1` are the sets `{1,4}` and `{2,3}`.

```
> p1:={{1,4},{2,3}};
> #p1;
2
> P:=p1^G;
> P;
GSet{
    {
        { 1, 4 },
        { 2, 3 }
    },
    {
        { 1, 2 },
        { 3, 4 }
    },
    {
        { 1, 3 },
        { 2, 4 }
    }
}
```

The set `P` has three elements; they are `p1`, `p2` and `p3`, where `p2` and `p3` are `{{2,4},{1,3}}` and `{{3,4},{1,2}}`.

```
> #P;
3
> p2:={{2,4},{1,3}};
> p3:={{3,4},{1,2}};
> P eq {p1,p2,p3};
true
> x:=G!(1,4,3,2);
```

```
> p1^x,p2^x,p3^x;
{
    { 3, 4 },
    { 1, 2 }
}
{
    { 1, 3 },
    { 2, 4 }
}
{
    { 1, 4 },
    { 2, 3 }
}
> p1^x eq p3, p3^x eq p1, p2^x eq p2;
true true true
```

Thus x gives rise to the permutation (p1,p3) of {p1,p2,p3}.

```
> f,L,K:=Action(G,P);
> P eq {p1,p2,p3};
true
> f;
Mapping from: GrpPerm: G to GrpPerm: L
> L;
Permutation group L acting on a set of cardinality 3
    ({
        { 1, 4 },
        { 2, 3 }
    }, {
        { 1, 2 },
        { 3, 4 }
    })
    ({
        { 1, 4 },
        { 2, 3 }
    }, {
        { 1, 3 },
        { 2, 4 }
    })
> K;
Permutation group K acting on a set of cardinality 4
Order = 4 = 2^2
    (1, 3)(2, 4)
    (1, 4)(2, 3)
> x:=G!(1,4,3,2);
> f(x);
({
```



```
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
})
> f(x) eq L!(p1,p3);
true
> f(G!(1,4));
({
    { 1, 2 },
    { 3, 4 }
}, {
    { 1, 3 },
    { 2, 4 }
})
> f(G!(2,4));
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
})
> f(G!(1,2,3));
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 3 },
    { 2, 4 }
}, {
    { 1, 2 },
    { 3, 4 }
})
> f(G!(1,2,4));
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
}, {
    { 1, 3 },
    { 2, 4 }
})
```

Thus the permutations $(1,4)$, $(2,4)$, $(1,2,3)$ and $(1,2,4)$ of $\{1,2,3,4\}$ give rise (respectively) to the permutations (p2,p3), (p1,p3), (p1,p2,p3) and (p1,p3,p2) of {p1,p2,p3}.

```
> Set(K);
{
    Id(K),
    (1, 3)(2, 4),
    (1, 2)(3, 4),
    (1, 4)(2, 3)
}
> f(G!(1,2)(3,4));
Id(L)
> f(G!(1,3)(2,4));
Id(L)
> f(G!(1,4)(2,3));
Id(L)
> A:={x*k : k in K};
> for t in A do f(t); end for;
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
})
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
})
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
})
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
})
```

So all the elements in the coset xK give rise to the same permutation of {p1,p2,p3}, namely (p1,p3).

```
> B:={G!(1,4)*k : k in K};
> for t in B do f(t); end for;
({
    { 1, 2 },
    { 3, 4 }
}, {
    { 1, 3 },
    { 2, 4 }
})
({
    { 1, 2 },
    { 3, 4 }
}, {
    { 1, 3 },
    { 2, 4 }
})
({
    { 1, 2 },
    { 3, 4 }
}, {
    { 1, 3 },
    { 2, 4 }
})
({
    { 1, 2 },
    { 3, 4 }
}, {
    { 1, 3 },
    { 2, 4 }
})
```

All the elements in the coset (1,4)K give rise to the same permutation of {p1,p2,p3}, namely (p2,p3).


```
> C:={G!(2,4)*k : k in K};
> for t in C do f(t);
for> end for;
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
})
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
})
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
})
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
})
> D:={G!(1,2,4)*k : k in K};
> for t in D do f(t);
for> end for;
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
}, {
    { 1, 3 },
    { 2, 4 }
})
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
}, {
    { 1, 3 },
    { 2, 4 }
})
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
}, {
    { 1, 3 },
    { 2, 4 }
})
({
    { 1, 4 },
    { 2, 3 }
}, {
    { 1, 2 },
    { 3, 4 }
}, {
    { 1, 3 },
    { 2, 4 }
})
```

All elements of (1,2,4)K give rise to (p1,p3,p2), and all elements of (2,4)K give rise to (p1,p3).

Why do elements of (2,4)K give rise to the same permutation of {p1,p2,p3} as do elements of xK = (1,4,3,2)K? Because (2,4)K = (1,4,3,2)K:

```
> G!(2,4)*K eq x*K;
true
```