An explicit string bundle

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Bundles on spheres

Recall:

$$\begin{aligned}\{SO(k)\text{-bundles on } S^n\} &\simeq [S^n, BSO(k)] \\ &\simeq [S^{n-1}, SO(k)] \\ &\simeq \pi_{n-1}(SO(k))\end{aligned}$$

Whitehead towers 1

For X a (path-connected) space, a *Whitehead tower* is a sequence

$$\cdots \to X_4 \to X_3 \to X_2 \to X_1 \to X$$

such that

X_n is n-connected

$\pi_i(X_n) \to \pi_i(X)$ is an isomorphism for $i > n$.

Example:

$X = SO(k)$

$$\cdots \to (**) \to Spin(k) \to Spin(k) \to SO(k)$$

Lifting the structure group

Recall that a k-manifold M is *spin* if the structure group of its tangent bundle can be lifted to $Spin(k)$.
The obstruction to this happening is a certain cohomology class in $H^2(M, \mathbb{Z}/2)$.

S^5

The frame bundle FS^5 is an $SO(5)$-bundle, classified by the nontrivial element in $\pi_4(SO(5)) = \mathbb{Z}/2$.

$H^2(S^5, \mathbb{Z}/2) = 0$, hence S^5 is spin.

The lift FS^5_{spin} of FS^5 to a $Spin(5)$-bundle is classified by

$$S^4 \xrightarrow{\eta} S^3 \simeq Sp(1) \hookrightarrow Sp(2) \simeq Spin(5)$$

so in fact lifts to an $Sp(1) = SU(2) = Spin(3)$-bundle.

Transition function

Recall that $S^4 \simeq \mathbb{HP}^1$, with homogeneous coordinates $[p; q]$, $p, q \in \mathbb{H}$ not *both* zero.

Proposition

The transition function $T\colon \mathbb{HP}^1 \to Sp(1)$ for FS^5_{spin} is given by

$$T[p;q] = \frac{2p\bar{q}i\bar{p}q - |p|^4 + |q|^4}{|p|^4 + |q|^4}$$

(ignoring irrelevant factors of $(-1, 1)$ for this talk)

Whitehead towers 2

- Q In

$$\cdots \to (**) \to Spin(k) \to Spin(k) \to SO(k)$$

  What is '$(**)$'?

- A1 A 3-connected topological group
- A2 *Not* a finite dimensional Lie group! (Hint: $\Rightarrow \pi_3(G) \neq 0$)
- A3 A Lie *2-group*, $String(k)$, presented by a *crossed module* of Lie groups (Baez-Crans-Schreiber-Stevenson arXiv:math/0504123).

Crossed modules

Definition

A *crossed module* is

- a map $t\colon K \to H$
- an action $a\colon H \times K \to K$
- such that t is H-equivariant and $a \circ (t \times id) = Ad$.

Key example: $String(k) = (\widehat{\Omega G} \to PG)$, for $G = Spin(k)$ (or even just a compact Lie group).

We will take $G = Sp(1)$. Then $\widehat{\Omega Sp(1)} \to PSp(1)$ is a model for $String(3)$.

(NB: more on $\widehat{\Omega G}$ later)

Cocycles 1

Recall the definition of a G-valued cocycle: an open cover $\{U_i\}$ of the space in question, and functions

$$g_{ij} \colon U_{ij} := U_i \cap U_j \to G \qquad \forall i, j$$

such that on U_{ijk}:

$$g_{ij} g_{jk} = g_{ik}$$

Cocycles 2

Let $t\colon K \to H$ be a crossed module such that $H \to H/t(K) = G$ has local sections. Say we want to lift the cocycle $\{g_{ij}\}$ to a $(K \to H)$-valued cocycle.

- We take an open cover $\coprod_\alpha U_{ij}^\alpha \to U_{ij}$ and functions $h_{ij}^\alpha\colon U_{ij}^\alpha \to H$ lifting g_{ij}.
- The h_{ij}^α only satisfy the cocycle condition on

$$U_{ijk}^{\alpha\beta\gamma} = U_{ij}^\alpha \cap U_{jk}^\beta \cap U_{ik}^\gamma$$

up to a $t(K)$-valued function, which we then lift to a function

$$k_{ijk}^{\alpha\beta\gamma}\colon U_{ijk}^{\alpha\beta\gamma} \to K$$

(if possible!)

Cocycles 3

The collection of functions $\{h_{ij}^{\alpha}, k_{ijk}^{\alpha\beta\gamma}\}$ satisfy a pair of equations (Breen 1994, Bartels arXiv:math/0410328, Baez-Schreiber arXiv:math/0511710):

$$h_{ij}^{\alpha} h_{jk}^{\beta} = t(k_{ijk}^{\alpha\beta\gamma}) h_{ik}^{\gamma} \qquad (C1)$$

and

$$a(h_{ij}^{\alpha}, k_{jkl}^{\beta\eta\varepsilon}) k_{ijl}^{\alpha\varepsilon\delta} = k_{ijk}^{\alpha\beta\gamma} k_{ikl}^{\gamma\eta\delta}, \qquad (C2)$$

which only a physicist could love.

Lifting the structure (2-)group

A k-manifold M is *string* if we can lift the transition functions for TM to a $(\Omega \widehat{Spin(k)} \to PSpin(k))$-valued cocycle. This is equivalent to the vanishing of a certain cohomology class in $H^4(M, \mathbb{Z})$ (cf Killingback, Witten 1980s, Stolz-Teichner 2004, Baez-Stevenson arXiv:0801.3843).

S^5 again

Since S^5 is so simple, we can trim down $(C1)$ and $(C2)$.

- The cover $\{U_i\}$ is $\{D^5_\pm\}$
- we take the cover $\{\mathbb{H}_\pm\}$ of $D^5_+ \cap D^5_- = \mathbb{HP}^1$ (ignoring factors of $(-1,1)$)

with the cocycle determined by functions

- $\tilde{T}_\pm \colon \mathbb{H}_\pm \to PSp(1)$, lifting T, and
- $T_{\widehat{\Omega}} \colon \mathbb{H}^\times \to \widehat{\Omega Sp(1)}$,

satisfying

- $\tilde{T}_-(q) = \tilde{T}_+(q) t(T_{\widehat{\Omega}}(q))$

Transition ‘function(s)’ 1

Using special charts on $Sp(1)$, can show the functions

$$\tilde{T}_+(q) = \left(s \mapsto \frac{|q|^4 - s^2 + 2\overline{q}iqs}{|q|^4 + s^2} \right),$$

$$\tilde{T}_-(p) = \left(s \mapsto \frac{|p|^4 s^2 - 1 + 2\overline{p}ips}{|p|^4 s^2 + 1} \cdot \left(\frac{s-i}{s+i}\right)^2 \right),$$

where $q \in \mathbb{H}_+,\ p \in \mathbb{H}_-$, lift T.

Then $T_\Omega(q) := \tilde{T}_+(q)^{-1}\,\tilde{T}_-(q^{-1})\colon \mathbb{H}^\times \to \Omega Sp(1)$ is

$$T_{\widehat{\Omega}}(q) = \left(s \mapsto \frac{(s+\overline{q}iq)(s\overline{q}iq - 1)}{(s-\overline{q}iq)(s\overline{q}iq + 1)} \cdot \left(\frac{s-i}{s+i}\right)^2 \right).$$

Transition 'function(s)' 2

Easiest description of $\widehat{\Omega Sp(1)}$ is as the cokernel of the homomorphism

$$\widetilde{\Omega^2 Sp(1)} \to P\Omega Sp(1) \ltimes U(1).$$

Note that we have then a map

$$P\mathbb{H}^\times \to P\Omega Sp(1) \to P\Omega Sp(1) \ltimes U(1) \twoheadrightarrow \widehat{\Omega Sp(1)},$$

which descends to $\mathbb{H}^\times$ as the latter is 2-connected.
Thus

$$T_{\widehat{\Omega}}(q) = [T_\Omega(q_t); 1] \in \widehat{\Omega Sp(1)}$$

for q_t any (polynomial, say) path in $\mathbb{H}^\times$ from 1 to q.

Transition 'function(s)' 3

Proposition

The functions

$$\tilde{T}_+(q) = \left(s \mapsto \frac{|q|^4 - s^2 + 2\overline{q}iqs}{|q|^4 + s^2} \right),$$

$$\tilde{T}_-(p) = \left(s \mapsto \frac{|p|^4 s^2 - 1 + 2\overline{p}ips}{|p|^4 s^2 + 1} \cdot \left(\frac{s-i}{s+i} \right)^2 \right),$$

$$T_{\widehat{\Omega}}(q) = \left[s \mapsto \frac{(s + \overline{q_t}iq_t)(s\overline{q_t}iq_t - 1)}{(s - \overline{q_t}iq_t)(s\overline{q_t}iq_t + 1)} \cdot \left(\frac{s-i}{s+i} \right)^2 ; 1 \right]$$

define the nontrivial $(\Omega\widehat{Spin(3)} \to PSpin(3))$-valued cocycle on S^5.

fin.