Fully nonlinear curvature flow of axially symmetric surfaces with boundary conditions

Fatemah Mofarreh

Supervisd by James McCoy and Graham Williams

University of Wollongong

1 October 2013

Talk outline

Axially symmetric initial hypersurface $X : [0, a] \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$,

$$X = (x, u(x)\,\omega),$$

$u : [0, a] \to \mathbb{R}^+$ axial graph function. Curvatures

$$\kappa_1 = \frac{-u_{xx}}{\left(1 + u_x^2\right)^{\frac{3}{2}}} = \frac{-\left(\arctan u_x\right)_x}{\sqrt{1 + u_x^2}}, \quad \kappa_2 = \ldots = \kappa_n = \frac{1}{u\sqrt{1 + u_x^2}}.$$

Normal curvature flow

$$\left(\frac{\partial X}{\partial t}(x, t)\right)^{\perp} = -F\left(\mathcal{W}(x, t)\right)\nu(x, t), \qquad (1)$$

equivalent to graph evolution for $u : [0, a] \times (0, T)$

$$\frac{\partial u}{\partial t} = -\sqrt{1 + u_x^2}\, F(\mathcal{W}). \qquad (2)$$

Example

Speed $F = H = \kappa_1 + \ldots + \kappa_n$, mean curvature flow (MCF).

- (Huisken, '90) MCF, pure Neumann boundary conditions, $H(\cdot, 0) \geq 0$. Curvature singularity at $T < \infty$ is *Type I*:

$$|A|^2 = \kappa_1^2 + \ldots + \kappa_n^2 \leq \frac{C}{T - t}.$$

- (Dzuik-Kawohl, '91) MCF, Neumann/mixed boundary conditions, $u(\cdot, 0), u_x(\cdot, 0) \geq 0$, $H(\cdot, 0) \geq 0$. Pinch off at $x = 0$ at $T < \infty$.
- (Matioc, '07) MCF, more general boundary conditions:

$$u_x(0, t) = 0 \text{ and either } u_x(a, t) = g(t) \text{ or } u(a, t) = h(t)$$

  where g and h satisfy some natural conditions, $u(\cdot, 0), u_x(\cdot, 0) \geq 0$, $H(\cdot, 0) \geq 0$. As $t \to T < \infty$, either

$$\kappa_2^2(0, t) \to \infty \text{ or } \kappa_1^2(a, t) \to \infty.$$

- (Escher-Matioc, '10) Periodic MCF, $u(\cdot, 0), u_x(\cdot, 0) \geq 0$ and $H > 0$ initially, single point pinch off at $T < \infty$.

Our speed $F(\mathcal{W}) = f(\kappa_1, \ldots, \kappa_n)$ satisfies

i) f smooth, symmetric function defined on $\mathbb{R}^n$.
ii) $\frac{\partial f}{\partial \kappa_i} > 0$ for each $i = 1, \ldots, n$ at every point of Γ.
iii) $f(k\kappa) = k\, f(\kappa)$ for any $k > 0$.
iv) $f(1, \ldots, 1) = 1$.
v) f is convex.

For partial singularity classification, need also

$$\lim_{z \to -\infty} f(z, 1, \ldots, 1) < 0.$$

Examples

$$f = \left(n + \eta\, n^{\frac{1}{p}}\right)^{-1} \left[\sum_{i=1}^{n} \kappa_i + \eta \left(\sum_{j=1}^{n} \kappa_j^p\right)^{\frac{1}{p}}\right]$$

for constants $\eta \in [0, 1)$ and $p \geq 1$.

Theorem (Short-time existence)

Given $u_0 \in C^2([0, a])$ *compatible with the boundary conditions, there exists* $\delta > 0$ *such that there is a unique solution* $u \in C^2([0, a] \times [0, \delta))$ *to* (2)*.*

Remarks:

1. Uniform parabolicity is not required, Condition ii) suffices.
2. Higher regularity (smoothing) *for a short time*, is standard.
3. We need C^2 initial data for our partial singularity classification.

We focus on the Neumann boundary conditions

$$u_x\left(0,t\right)=0, u_x\left(a,t\right)=g\left(t\right) \tag{3}$$

where g is smooth, non-negative and non-increasing.

Lemma

Under the flow (1)*,*

1. $u_x\left(\cdot,0\right)\geq 0 \implies u_x\left(\cdot,t\right)\geq 0$;
2. $u_t\left(\cdot,0\right)\leq 0$ *that is,* $F\left(\mathcal{W}\left(\cdot,0\right)\right)\geq 0 \implies u_t\left(\cdot,t\right)\leq 0.$

Idea of Proof: Under (2), $v=u_x$ and $v=u_t$ satisfy

$$\frac{\partial}{\partial t}v=\frac{\dot{f}^1}{1+u_x^2}v_{xx}-\frac{2\dot{f}^1}{\left(1+u_x^2\right)^2}u_x u_{xx} v_x+\sum_{j=2}^{n}\dot{f}^j\frac{v}{u^2}.$$

Results follow by the maximum principle, and Hopf Lemma on boundary. □

Lemma

If, in addition to $u_0 > 0, (u_x)_0 \geq 0$ *and* $F(\mathcal{W}_0) \geq 0$*, we have*

$$\arctan g(0) < \frac{(n-1)a^2}{\int_0^a u_0(x)dx}, \qquad \text{then } T < \infty.$$

Idea of Proof: If instead $T = \infty$ then $u > 0$ for all t. Set $E(t) = \int_0^a u(x,t)\,dx$. Then

$$E'(t) = \int_0^a \frac{\partial}{\partial t} u\,dx = -\int_0^a \sqrt{1 + u_x^2} F(\mathcal{W})\,dx.$$

Since F is convex, $F \geq \frac{1}{n} H$ so

$$E'(t) \leq -\frac{1}{n}\int_0^a \sqrt{1 + u_x^2} H dx \leq \frac{1}{n}\int_0^a (\arctan u_x)_x\,dx - \frac{n-1}{n}\int_0^a \frac{1}{u} dx.$$

It follows using Hölder's inequality and the assumption that $E'(t) < 0$, contradicting $E(t) > 0$ for all t. Hence $T < \infty$. □

Theorem (McCoy, Mofarreh, Williams '13)

Suppose F satisfies given conditions and (2) is accompanied by boundary conditions (3) and $u_0 > 0, (u_0)_x \geq 0$ and $F \geq 0$.

If $\lim_{t \to T} \kappa_1^2 (a, T) < \infty$ then $\lim_{t \to T} \kappa_j^2 (0, t) = \infty$ for $j = 2, \dots, n$.

Idea of Proof:

- Suppose instead $\lim_{t \to T} \kappa_1^2 (a, T) < \infty$ and $\lim_{t \to T} \kappa_j^2 (0, T) < \infty$.
- We show solution could be extended beyond $t = T$.
- So we need to show that κ_1^2 and κ_j^2 remain bounded up to $t = T$, so $u(\cdot, T) \in C^2([0, a])$ could be used as new u_0.
- By assumption $\lim_{t \to T} u(0, t) = \delta > 0$ so

$$\kappa_j^2 (x, t) = u(x, t)^{-2} \left[1 + u_x(x, t)^{-2}\right] \leq u(x, t)^{-2} \leq \delta^{-2}.$$

To bound κ_1^2 we need to bound u_{xx} from above and from below.

u_{xx} bound above

By homogeneity, since $\kappa_2 > 0$,

$$f(\kappa_1, \kappa_2, \ldots, \kappa_2) = \kappa_2 f\left(\frac{\kappa_1}{\kappa_2}, 1, \ldots\right) \geq 0$$

This implies

$$f(z, 1, \ldots, 1) \geq 0$$

for $z = \frac{\kappa_1}{\kappa_2}$. Condition on f then implies

$$z = \frac{\kappa_1}{\kappa_2} = \frac{-u\, u_{xx}}{1 + u_x^2} \geq -c_0 \implies \frac{u_{xx}}{1 + u_x^2} \leq \frac{c_0}{u} \leq \frac{c_0}{\delta}.$$

u_{xx} **bound below** The function $w = -u_t\, e^{-\lambda t}$ satisfies

$$\frac{\partial}{\partial t} w = \frac{\dot{f}^1}{1+u_x^2} w_{xx} - \frac{2\dot{f}^1}{\left(1+u_x^2\right)^2} u_x u_{xx} w_x + \left(\sum_{j=2}^{n} \frac{\dot{f}^j}{u^2} - \lambda \right) w.$$

Using the maximum principle, convexity of F and the boundary conditions we find that for $\lambda > \delta^{-2}$,

$$w = -u_t\, e^{-\lambda t} = \sqrt{1+u_x^2}\, F\, e^{-\lambda t} \leq \overline{C}\left(M_0, \delta, T\right)$$

and so

$$\frac{1}{n}\sqrt{1+u_x^2}\, H \leq \sqrt{1+u_x^2} F \leq \overline{C} e^{\lambda T}.$$

Therefore

$$\frac{-u_{xx}}{1+u_x^2} + \frac{n-1}{u} \leq n\, \overline{C}\, e^{\lambda T}$$

providing the lower bound on u_{xx}.

So we have $\kappa_1^2 \leq C\left(M_0, \delta, T\right)$ as required. □

Now consider, for $0 < k \leq 1$ constant, the flow

$$\frac{\partial u}{\partial t} = -\sqrt{1 + u_x^2}\, F^k,$$

where F satisfies the same conditions.

Theorem (McCoy, Mofarreh, Williams ’13)

In the case of pure Neumann boundary conditions and

$$u_0 > 0, (u_0)_x \geq 0 \text{ and } F > 0,$$

if $\lim_{t \to T} \kappa_1^2 (a, T) < \infty$ *then* $\lim_{t \to T} \kappa_j^2 (0, t) = \infty$ *for* $j = 2, \ldots, n$.

Remarks on pure Neumann boundary conditions:

1. Allow comparison with cylinders to show that $T < \infty$.
2. Allow application of the maximum principle to the periodic solution to obtain $F > 0$ holds under the flow (not just $F \geq 0$); needed in the proof for $0 < k < 1$.

DK Dzuik, G, Kawohl, B, *On rotationally symmetric mean curvature flow*, J. Differential Equations **93** 1 (1991), 142–150.

EM Escher, J, Matioc, B-V, *Neck pinching for periodic mean curvature flows*, Analysys **30** (2010), 253–260.

Hu Huisken, G, *Asymptotic behaviour for singularities of the mean curvature flow*, J. Differential Geometry **31** (1990), 285–299.

M Matioc, B-V, *Boundary value problems for rotationally symmetric mean curvature flows*, Arch. Math. **89** (2007), 365–372.

MMW McCoy, J A, Mofarreh, F Y Y, Williams, G H, *Fully nonlinear curvature flow of axially symmetric hypersurfaces with boundary conditions*, Annali di Matematica, (2013), 1-13.