Universal Capelli identities and quantum immanants for $\mathfrak{q}_N$

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- Sergeev duality and Schur Q-polynomials

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Joint work with Iryna Kashuba `arXiv:2512.21631`

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Then $\mathcal{E}_{\mathcal{U}}(\mathbb{C}^N)^{\otimes n} \cong L(\lambda)$ as $\mathfrak{gl}_N$-modules. Hence,

$$\operatorname{tr} \mathcal{E}_{\mathcal{U}} Y_1 \ldots Y_n = s_\lambda(y_1, \ldots, y_N),$$

where $Y = \operatorname{diag}(y_1, \ldots, y_N)$ and $Y_a = 1^{\otimes(a-1)} \otimes Y \otimes 1^{\otimes(n-a)}$.

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Consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{N|N}$ with the canonical basis $e_{-N}, \dots, e_{-1}, e_1, \dots, e_N$ with $p(e_i) = \bar{\imath} \mod 2$, where

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The Sergeev superalgebra $\mathcal{S}_n = \mathbb{C}\mathfrak{S}_n \ltimes \mathcal{C}l_n$ acts on the space

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$$J = \sum_{i=-N}^{N} E_{i,-i}(-1)^{\bar{\imath}} \in \operatorname{End} \mathbb{C}^{N|N}.$$

The superalgebra $Q_N \subset \operatorname{End} \mathbb{C}^{N|N}$ has the basis

$$e_{kl} = E_{kl} + E_{-k,-l}, \qquad f_{kl} = E_{k,-l} + E_{-k,l}, \qquad 1 \leqslant k, l \leqslant N,$$

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and $Q_\lambda(y_1, \dots, y_N)$ is the Schur Q-polynomial.

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$1'$	1	$2'$	2	2	2
	2	$3'$	3		
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The factorial Schur Q-polynomial is given by

$$Q_\lambda^+(y) = \sum_{\mathrm{sh}(\mathcal{T})=\lambda} \prod_{\alpha\in\lambda} \big(y_{\mathcal{T}(\alpha)} + \mathrm{sgn}\,(\mathcal{T}(\alpha))\sigma(\alpha)\big),$$

where $\sigma(\alpha) = j - i$ is the content of $\alpha = (i, j)$ [Ivanov, 2005].

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Vanishing property:

$$Q^+_\lambda(-\mu) = 0$$

for all strict μ with $\ell(\mu) \leqslant N$ and $|\mu| < |\lambda|$.

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Characterization property [Ivanov, 2005]:

If the top degree component of a polynomial $P(y) \in \Gamma_N$ coincides with $Q_\lambda(y)$ for some λ with $\ell(\lambda) \leqslant N$, and $P(-\mu) = 0$ for all strict μ with $\ell(\mu) \leqslant N$ and $|\mu| < |\lambda|$, then $P(y) = Q^+_\lambda(y)$.

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Lie superalgebras $\mathfrak{q}_N$ and $\mathfrak{q}_M$ act on $\mathcal{P}$; for $\mathfrak{q}_N$ we have

$$e_{kl} \mapsto \sum_{a=-M}^{M} x_{ak} \partial_{la},$$

$$f_{kl} \mapsto \sum_{a=-M}^{M} x_{ak} \partial_{l,-a},$$

with $k, l = 1, \dots, N$.

We have a decomposition for polynomials of degree n:

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[Cheng and Wang, 2000, Sergeev, 2001].

In particular, the highest weight $\mathfrak{q}_N$-modules $L(\lambda)$ can be realized as submodules of $\mathcal{P}^n$ if $\ell(\lambda) \leqslant M$ and $\ell(\lambda) \leqslant N$.

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We will twist the action of $\mathfrak{q}_N$ with the automorphism

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This defines another action of $\mathfrak{q}_N$ in $\mathcal{P}$,

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We thus get a homomorphism $\mathrm{U}(\mathfrak{q}_N) \to \mathcal{P}\mathcal{D}$.

Combine the variables and derivations into matrices X and D:

$$X = \sum_{a=1}^{M} \sum_{k=1}^{N} \left(e_{ka} \otimes x_{ak} - i f_{ka} \otimes x_{-a,k} \right) \in \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^{N|N}) \otimes \mathcal{PD}$$

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$$D = \sum_{a=1}^{M} \sum_{k=1}^{N} \left(i\, e_{ak} \otimes \partial_{k,-a} + f_{ak} \otimes \partial_{ka}\right) \in \mathrm{Hom}(\mathbb{C}^{N|N}, \mathbb{C}^M) \otimes \mathcal{PD},$$

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Introduce the odd element

$$G = \sum_{k,l=1}^{N} \left(e_{kl} \otimes f_{lk} - f_{kl} \otimes e_{lk} \right) \in Q_N \otimes \mathrm{U}(\mathfrak{q}_N).$$

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and for $b = 1, \dots, n$ set

$$M^{(b)} = T_{1b} + \dots + T_{b-1,b} \in (Q_N)^{\otimes n}.$$

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This is the image of the odd Jucys–Murphy element.

Theorem. Under the action $G \mapsto XD$, we have

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Proof. Simple induction argument; suppose $n = 2$.

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$$\widetilde{T}_{12} = \sum_{a=1}^{M} \sum_{l=1}^{N} \big(f_{al} \otimes e_{la} - e_{al} \otimes f_{la}\big).$$

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$$D_1 X_2 = X_2 D_1 + \widetilde{T}_{12},$$

where

$$\widetilde{T}_{12} = \sum_{a=1}^{M} \sum_{l=1}^{N} \big(f_{al} \otimes e_{la} - e_{al} \otimes f_{la} \big).$$

The desired identity now follows from the relation

$$\widetilde{T}_{12} D_2 + D_1 T_{12} = 0.$$

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Modified matrices X and D; they are even:

$$X = \sum_{a=1}^{M} \sum_{k=1}^{N} \big((E_{ka} + E_{-k,-a}) \otimes x_{ak} - i\, (E_{-k,a} + E_{k,-a}) \otimes x_{-a,k} \big),$$

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and

$$D = \sum_{a=1}^{M} \sum_{k=1}^{N} \big((E_{-a,-k} - E_{ak}) \otimes \partial_{ka} + i\,(E_{-a,k} - E_{a,-k}) \otimes \partial_{k,-a} \big).$$

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be the image of the even Jucys–Murphy element x_b under the action of $\mathcal{S}_n$ on the tensor product space $(\mathbb{C}^{N|N})^{\otimes n}$.

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Theorem. Under the action $F \mapsto XD$, we have

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for the image of the element of $(\mathrm{End}\,\mathbb{C}^{N|N})^{\otimes n} \otimes \mathrm{U}(\mathfrak{q}_N)$.

Corollary. Under the action $F \mapsto XD$, we have

$$\big(F_1 + \kappa_1(\mathcal{U})\big) \dots \big(F_n + \kappa_{\mathsf{n}}(\mathcal{U})\big)\, \mathcal{E}_{\mathcal{U}} \mapsto X_1 \dots X_n D_1 \dots D_n\, \mathcal{E}_{\mathcal{U}},$$

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the signed contents are defined by

$$\kappa_{\mathsf{a}}(\mathcal{U}) = \begin{cases} \sqrt{\sigma_a(\sigma_a + 1)} & \text{if } \mathsf{a} = a \text{ is unbarred,} \\ -\sqrt{\sigma_a(\sigma_a + 1)} & \text{if } \mathsf{a} = \bar{a} \text{ is barred,} \end{cases}$$

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Proof. $\mathcal{X}^{(n)}\mathcal{E}_{\mathcal{U}} = \kappa_{\mathsf{n}}(\mathcal{U})\, \mathcal{E}_{\mathcal{U}}$.

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Proof. $\mathcal{X}^{(n)}\mathcal{E}_{\mathcal{U}} = \kappa_{\mathsf{n}}(\mathcal{U})\, \mathcal{E}_{\mathcal{U}}$. Cf. [Okounkov, 1996] for $\mathfrak{gl}_N$.

Quantum immanants

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The quantum immanant associated with $\lambda \Vdash n$ is defined by

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Moreover, it depends only on λ and does not depend on $\mathcal{U}$.

Recall the Harish-Chandra isomorphism [Sergeev, 1999]:

$$\chi : \mathrm{Z}(\mathfrak{q}_N) \to \Gamma_N.$$

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On any highest vector $\xi \in L(y)$ we have

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The scalar is a supersymmetric polynomial denoted by $\chi(\mathbb{S}_\lambda)$.

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By the corollary, under the action $F \mapsto XD$, we have

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Taking into account the twisting of the action of $\mathfrak{q}_N$ coming from the queer Howe duality, the element $\mathbb{S}_{\lambda}$ acts as zero in all $L(-\widetilde{\mu})$ with $|\mu| < |\lambda| = n$, where $\widetilde{\mu} = (\mu_N, \dots, \mu_1)$.

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for the column tableau $\mathcal{U}^c$. We can show that for any $\mathcal{U}$,

$$\frac{g_\lambda}{n!}\, C_\lambda = \operatorname{str} \mathcal{E}_{\mathcal{U}} \left(F_1 - \kappa_1(\mathcal{U})\right) \dots \left(F_n - \kappa_{\mathsf{n}}(\mathcal{U})\right).$$

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Under the anti-automorphism $F \to -F$, this maps to $(-1)^n \mathbb{S}_\lambda$.

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with the Harish-Chandra images in *loc. cit.* corrected by changing the signs of the variables.

Appendix. Spin symmetric group algebra

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This is the superalgebra $\mathbb{C}\mathfrak{S}_n^-$ generated by odd elements

$t_1, \dots, t_{n-1}$ subject to the relations

$$t_a^2 = 1, \qquad t_a t_{a+1} t_a = t_{a+1} t_a t_{a+1}, \qquad t_a t_b = -t_b t_a, \quad |a-b| > 1.$$

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The transpositions are defined by

$$t_{ab} = (-1)^{b-a-1}\, t_{b-1} \dots t_{a+1} t_a t_{a+1} \dots t_{b-1}, \qquad a < b,$$

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The odd Jucys–Murphy elements are

$$m_1 = 0, \qquad m_a = t_{1a} + \dots + t_{a-1,a}, \qquad a = 2, \dots, n.$$

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where

$$s_a = \frac{1}{\sqrt{2}}\, t_a (c_{a+1} - c_a).$$

Simple modules over $\mathcal{S}_n$ and $\mathbb{C}\mathfrak{S}_n^-$ are parameterized by strict partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n with $\lambda_1 > \cdots > \lambda_\ell > 0$ and $\lambda_1 + \cdots + \lambda_\ell = n$.

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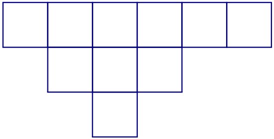
Depict λ by the shifted Young diagram: e.g. for $\lambda = (6, 3, 1)$



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The dimensions of the simple modules over $\mathcal{S}_n$ and $\mathbb{C}\mathfrak{S}_n^-$:

$$\dim U^\lambda = 2^{n-\lfloor\frac{\ell(\lambda)}{2}\rfloor} g_\lambda \qquad \text{and} \qquad \dim V^\lambda = 2^{\lceil\frac{n-\ell(\lambda)}{2}\rceil} g_\lambda,$$

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where g_λ is the number of standard λ-tableaux, found by the Schur formula [1911] (see also [Morris 1965] for hook-length formula):

$$g_\lambda = \frac{n!}{\lambda_1! \dots \lambda_\ell!} \prod_{1 \leqslant i < j \leqslant \ell} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$