Representations of the odd symmetric group

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Plan

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- Review of representations of the symmetric group

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- Spin symmetric group algebra and Sergeev superalgebra

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Overlapping with independent work by Shuo Li and Lei Shi

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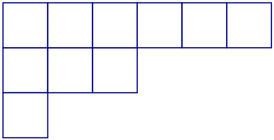
Identify λ with its Young diagram; e.g. $\lambda = (6,3,1)$ is drawn as



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In the example, $\sigma_6(\mathcal{T}) = 0$, $\sigma_7(\mathcal{T}) = -2$ and $\sigma_8(\mathcal{T}) = 4$.

For $\lambda \vdash n$ the module V^λ over $\mathfrak{S}_n$ is afforded by the vector space

$$V^\lambda = \bigoplus_{\mathrm{sh}(\mathcal{T})=\lambda} \mathbb{C}\, v_{\mathcal{T}}$$

with the basis vectors $v_{\mathcal{T}}$ labelled by the standard λ-tableaux $\mathcal{T}$.

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The generators $s_a = (a, a+1)$ and the JM elements x_a act by

$$s_a v_{\mathcal{T}} = \frac{1}{\sigma_{a+1}(\mathcal{T}) - \sigma_a(\mathcal{T})} v_{\mathcal{T}} + \mathcal{X}_a(\mathcal{T})\, v_{s_a\mathcal{T}},$$

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where $v_{s_a\mathcal{T}} := 0$ if the tableau $s_a\mathcal{T}$ is not standard, and

$$\mathcal{X}_a(\mathcal{T}) = \sqrt{1 - \frac{1}{\big(\sigma_{a+1}(\mathcal{T}) - \sigma_a(\mathcal{T})\big)^2}}.$$

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Define by induction,

$$e_{\mathcal{U}} = e_{\mathcal{V}} \frac{(x_n - a_1)\dots(x_n - a_l)}{(c - a_1)\dots(c - a_l)},$$

where $a_1, \dots, a_l$ are the contents of all addable boxes of $\mathrm{sh}(\mathcal{V})$ except for α.

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$$e_{\mathcal{U}} e_{\mathcal{V}} = \delta_{\mathcal{U}\mathcal{V}} e_{\mathcal{V}}, \qquad 1 = \sum_{\lambda \vdash n} \sum_{\mathrm{sh}(\mathcal{U})=\lambda} e_{\mathcal{U}}.$$

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Proof. Observe that the $\mathbb{C}\mathfrak{S}_n$-module $V = \bigoplus_{\lambda \vdash n} V^{\lambda}$ is faithful. All relations follow from the action: $e_{\mathcal{U}}\, v_{\mathcal{T}} = \delta_{\mathcal{U}\mathcal{T}}\, v_{\mathcal{T}}$.

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and have the properties

$$\phi_a^2 = 1 - (x_a - x_{a+1})^2.$$

For each $w \in \mathfrak{S}_n$ there are well-defined elements

$$\phi_w = \phi_{a_1} \dots \phi_{a_r} \qquad \text{and} \qquad \phi_w^* = \phi_{w^{-1}} = \phi_{a_r} \dots \phi_{a_1},$$

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Hence,

$$\phi_{d(\mathcal{T})} v_{\mathcal{R}^\lambda} = b_{\mathcal{T}} v_{\mathcal{T}}, \qquad b_{\mathcal{T}} \neq 0,$$

where $\mathcal{R}^\lambda$ is the row-tableau of shape λ and $\mathcal{T} = d(\mathcal{T}) \mathcal{R}^\lambda$.

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Theorem (Murphy 1992). As λ runs over the Young diagrams with n boxes, the elements $\zeta_{\mathcal{T}\mathcal{U}}$ associated with standard λ-tableaux $\mathcal{T}$ and $\mathcal{U}$ form a basis of the group algebra $\mathbb{C}\mathfrak{S}_n$.

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Apply the linear combination to the basis vector $v_{\mathcal{W}} \in V$.

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Proof. Note that $\zeta_{\mathcal{T}\mathcal{V}} e_{\mathcal{U}} = \delta_{\mathcal{V}\mathcal{U}} \zeta_{\mathcal{T}\mathcal{U}}$ so that the vectors $\zeta_{\mathcal{T}\mathcal{U}}$ form a basis of $\mathbb{C}\mathfrak{S}_n e_{\mathcal{U}}$.

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where every left ideal $\mathbb{C}\mathfrak{S}_n e_{\mathcal{U}}$ is a simple $\mathfrak{S}_n$-module associated with the partition $\lambda = \mathrm{sh}(\mathcal{U})$.

Fusion procedure

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Take n variables $u_1, \dots, u_n$ and consider the rational function with values in $\mathbb{C}\mathfrak{S}_n$ defined by

$$\Phi(u_1, \dots, u_n) = \prod_{1 \leqslant a < b \leqslant n} \Big(1 - \frac{(a,b)}{u_a - u_b}\Big),$$

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where the product is taken in the lexicographical order on the set of pairs (a, b).

Given a standard λ-tableau $\mathcal{U}$, set $\sigma_a = \sigma_a(\mathcal{U})$ for $a = 1, \ldots, n$.

Theorem (Jucys 1966; version of M. 2008). The consecutive evaluations are well-defined and we have

$$\Phi(u_1, \ldots, u_n)\big|_{u_1=\sigma_1}\big|_{u_2=\sigma_2} \ldots \big|_{u_n=\sigma_n} = \frac{n!}{f_\lambda}\, e_{\mathcal{U}}.$$

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Applications to representation theory of $\mathfrak{gl}_N$ and the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ use the fact that the factors in $\Phi(u_1,\dots,u_n)$ are Yang R-matrices satisfying the Yang–Baxter equation.

Spin symmetric group algebra

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This is the superalgebra $\mathbb{C}\mathfrak{S}_n^-$ generated by odd elements

$t_1, \ldots, t_{n-1}$ subject to the relations

$$t_a^2 = 1, \qquad t_a t_{a+1} t_a = t_{a+1} t_a t_{a+1}, \qquad t_a t_b = -t_b t_a, \quad |a-b| > 1.$$

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The transpositions are defined by

$$t_{ab} = (-1)^{b-a-1}\, t_{b-1} \dots t_{a+1} t_a t_{a+1} \dots t_{b-1}, \qquad a < b,$$

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where

$$s_a = \frac{1}{\sqrt{2}}\, t_a (c_{a+1} - c_a).$$

Simple modules over $\mathcal{S}_n$ and $\mathbb{C}\mathfrak{S}_n^-$ are parameterized by strict partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n with $\lambda_1 > \cdots > \lambda_\ell > 0$ and $\lambda_1 + \cdots + \lambda_\ell = n$.

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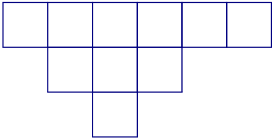
Depict λ by the shifted Young diagram: e.g. for $\lambda = (6, 3, 1)$



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The dimensions of the simple modules over $\mathcal{S}_n$ and $\mathbb{C}\mathfrak{S}_n^-$:

$$\dim U^\lambda = 2^{n-\lfloor \frac{\ell(\lambda)}{2} \rfloor} g_\lambda \qquad \text{and} \qquad \dim V^\lambda = 2^{\lceil \frac{n-\ell(\lambda)}{2} \rceil} g_\lambda,$$

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where g_λ is the number of standard λ-tableaux, found by the Schur formula [1911] (see also [Morris 1965] for hook-length formula):

$$g_\lambda = \frac{n!}{\lambda_1! \dots \lambda_\ell!} \prod_{1 \leqslant i < j \leqslant \ell} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

Both superalgebras are semisimple; the Wedderburn decompositions yield the Schur identity:

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Introduce the barred tableaux by allowing any non-diagonal entry to occur with a bar on it; cf. [Worley 1984], [Sagan 1987]:

$$\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 2 & \bar{4} & \bar{5} & 8 & \overline{10} \\
\hline
\multicolumn{1}{c|}{} & 3 & \bar{6} & 9 & \multicolumn{2}{c}{} \\
\cline{2-4}
\multicolumn{2}{c|}{} & 7 & \multicolumn{3}{c}{} \\
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\end{array}\;.$$

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		7			

.

Note that given a strict partition $\lambda \Vdash n$, the number of the corresponding standard barred tableaux equals $2^{n-\ell(\lambda)}\, g_\lambda$.

Given a standard barred tableaux $\mathcal{U}$, the signed content $\kappa_{\mathsf{a}}(\mathcal{U})$

of any barred or unbarred entry a of $\mathcal{U}$ is

$$\kappa_{\mathsf{a}}(\mathcal{U}) = \begin{cases} \sqrt{\sigma_a(\sigma_a+1)} & \text{if } \mathsf{a} = a \text{ is unbarred,} \\ -\sqrt{\sigma_a(\sigma_a+1)} & \text{if } \mathsf{a} = \bar{a} \text{ is barred,} \end{cases}$$

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The even Jucys–Murphy elements in $\mathcal{S}_n$ are [Nazarov 1997]:

$$x_a = \sqrt{2}\, m_a c_a = \sqrt{2}\,(t_{1a} + \cdots + t_{a-1,a})\, c_a, \qquad a = 1, \ldots, n.$$

Note that $x_a^2 = 2m_a^2$ and the x_a pairwise commute.

Idempotents in $\mathcal{S}_n$

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For any standard barred tableau $\mathcal{U}$ introduce the element $e_{\mathcal{U}}$ of $\mathcal{S}_n$ by induction, setting

$$e_{\mathcal{U}} = e_{\mathcal{V}} \, \frac{(x_n - b_1) \dots (x_n - b_p)}{(\kappa - b_1) \dots (\kappa - b_p)},$$

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Here $b_1, \dots, b_p$ are the signed contents in all addable boxes of $\mathrm{sh}(\mathcal{V})$ (barred and unbarred), except for the entry n (resp. $\bar{n}$), while κ is the signed content of the entry n (resp. $\bar{n}$).

Example. For $n = 2$ we have two barred tableaux $\mathcal{U}$:

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The respective elements $e_{\mathcal{U}} \in \mathcal{S}_3$ are

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we have

$$e_{\mathcal{U}} = \frac{\sqrt{2} - x_2}{2\sqrt{2}} \cdot \frac{(x_3 - \sqrt{6})(x_3 + \sqrt{6})}{-6} = \frac{\sqrt{2} - x_2}{2\sqrt{2}} \cdot \frac{6 - x_3^2}{6}.$$

Proposition. All elements $e_{\mathcal{U}}$ are idempotents in $\mathcal{S}_n$. They are pairwise orthogonal and form a decomposition of the identity:

$$e_{\mathcal{U}}\, e_{\mathcal{V}} = \delta_{\mathcal{U}\mathcal{V}}\, e_{\mathcal{V}}, \qquad 1 = \sum_{\lambda \Vdash n} \sum_{\mathrm{sh}(\mathcal{U})=\lambda} e_{\mathcal{U}}.$$

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Proof. Use the faithful $\mathcal{S}_n$-module

$$\widehat{U} = \bigoplus_{\lambda \Vdash n} \widehat{U}^{\lambda}.$$

Here

$$\widehat{U}^{\lambda} = \bigoplus_{\mathrm{sh}(\mathcal{T})=\lambda} \mathcal{C}l_n v_{\mathcal{T}}$$

with the basis vectors $v_{\mathcal{T}}$ labelled by standard λ-tableaux $\mathcal{T}$.

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The generators of $\mathcal{S}_n$ act by

$$s_a v_{\mathcal{T}} = \left(\frac{1}{\kappa_{a+1}(\mathcal{T}) - \kappa_a(\mathcal{T})} + \frac{c_a c_{a+1}}{\kappa_{a+1}(\mathcal{T}) + \kappa_a(\mathcal{T})} \right) v_{\mathcal{T}} + \mathcal{Y}_a(\mathcal{T})\, v_{s_a \mathcal{T}},$$

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Corollary. We have the direct sum decomposition

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summed over standard barred tableaux $\mathcal{U}$ of shape λ.

Moreover,

$$\mathcal{S}_n e_{\mathcal{U}} \cong \widehat{U}^{\lambda} \qquad \text{and} \qquad \widehat{U}^{\lambda} \cong \underbrace{U^{\lambda} \oplus \cdots \oplus U^{\lambda}}_{2^{\lfloor \frac{\ell(\lambda)}{2} \rfloor}}.$$

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Since

$$\mathcal{C}l_{2m} \cong (\mathcal{C}l_2)^{\otimes m} \qquad \text{and} \qquad \mathcal{C}l_{2m+1} \cong (\mathcal{C}l_2)^{\otimes m} \otimes \mathcal{C}l_1,$$

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there are 2^m pairwise orthogonal primitive idempotents in $\mathcal{C}l_{2m}$ and $\mathcal{C}l_{2m+1}$.

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Introduce the corresponding idempotents $\mathcal{E}_1^{\mathcal{U}}, \ldots, \mathcal{E}_{2^m}^{\mathcal{U}}$ in $\mathcal{C}l_\ell^{\mathcal{U}}$, where $m = \lfloor \frac{\ell(\lambda)}{2} \rfloor$.

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$$e_{\mathcal{U}}^{(r)} := \mathcal{E}_r^{\mathcal{U}} e_{\mathcal{U}} = e_{\mathcal{U}} \mathcal{E}_r^{\mathcal{U}},$$

with $r = 1, \ldots, 2^m$, are idempotents in $\mathcal{S}_n$.

Theorem. We have the direct sum decomposition

$$\mathcal{S}_n = \bigoplus_{\lambda \Vdash n} \bigoplus_{\mathrm{sh}(\mathcal{U})=\lambda} \bigoplus_{r=1}^{2^m} \mathcal{S}_n e_{\mathcal{U}}^{(r)}.$$

Theorem. We have the direct sum decomposition

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Moreover, the left ideal is an irreducible $\mathcal{S}_n$-module,

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Proof. It suffices to show that the left ideal has the right dimension. This follows from the Schur identity.

Murphy-type basis in $\mathcal{S}_n$

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Nazarov's intertwiners

$$\phi_a = s_a(x_a^2 - x_{a+1}^2) + x_a + x_{a+1} - c_a c_{a+1}(x_a - x_{a+1}).$$

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They satisfy the braid relations

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$$\phi_a^2 = 2(x_a^2 + x_{a+1}^2) - (x_a^2 - x_{a+1}^2)^2$$

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together with

$$\phi_a c_a = c_{a+1}\phi_a, \qquad \phi_a c_{a+1} = c_a\phi_a, \qquad \phi_a c_b = c_b\phi_a, \quad b \neq a, a+1.$$

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Let β be a set of non-diagonal boxes of λ. Denote by $\mathcal{R}^{\lambda,\beta}$ the tableau obtained from $\mathcal{R}^\lambda$ by adding the bars to all entries occupying the boxes of β.

For any two standard unbarred tableaux $\mathcal{T}$ and $\mathcal{U}$ of shape λ set

$$\zeta^{\beta}_{\mathcal{T}\mathcal{U}} = \phi_{d(\mathcal{T})}\, e_{\mathcal{R}^{\lambda,\beta}}\, \phi^{*}_{d(\mathcal{U})},$$

where $d(\mathcal{T}) \in \mathfrak{S}_n$ is such that $\mathcal{T} = d(\mathcal{T})\mathcal{R}^\lambda$.

Theorem. As λ runs over shifted Young diagrams with n boxes, the elements $\zeta^{\beta}_{\mathcal{T}\mathcal{U}}$ associated with standard unbarred tableaux $\mathcal{T}$ and $\mathcal{U}$ of shape λ and sets β of non-diagonal boxes of λ form a basis of the Sergeev superalgebra $\mathcal{S}_n$ over $\mathcal{C}l_n$.

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Proof. The number of elements is

$$\sum_{\lambda \Vdash n} 2^{n-\ell(\lambda)}\, g_{\lambda}^{2} = n!,$$

and so coincides with the rank of $\mathcal{S}_n$ as a $\mathcal{C}l_n$-module. It is enough to show that the elements are linearly independent over $\mathcal{C}l_n$ by acting in $\widehat{U}$.

Seminormal form for $\mathcal{S}_n$

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Recall the $\mathcal{S}_n$-module

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Theorem. The simple module U^λ over $\mathcal{S}_n$ is the quotient of $\widehat{U}^\lambda$ by the relations

$$c_{d_{2a}} v_{\mathcal{T}} = i\, c_{d_{2a-1}} v_{\mathcal{T}}, \qquad a = 1, \dots, \left\lfloor \frac{\ell(\lambda)}{2} \right\rfloor.$$

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Introduce the vectors $\theta_{\mathcal{T}} \in \widehat{U}^\lambda$ by

$$\theta_{\mathcal{T}} = y_{a_r} \dots y_{a_1} v_{\mathcal{T}}$$

with the use of a fixed reduced decomposition $d(\mathcal{T}) = s_{a_1} \dots s_{a_r}$.

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Lemma.

$$t_a \theta_{\mathcal{T}} = \frac{\sqrt{2}\left(\kappa_a(\mathcal{T})\, c_{a^{\mathcal{T}}} - \kappa_{a+1}(\mathcal{T})\, c_{(a+1)^{\mathcal{T}}}\right)}{\kappa_a(\mathcal{T})^2 - \kappa_{a+1}(\mathcal{T})^2}\, \theta_{\mathcal{T}} + \varepsilon_a(\mathcal{T})\, \mathcal{Y}_a(\mathcal{T})\, \theta_{s_a \mathcal{T}},$$

where $\varepsilon_a(\mathcal{T}) = \pm 1$.

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Key observation: The RHS depends only on the Clifford generators, corresponding to the non-diagonal entries of $\mathcal{R}^{\lambda}$.

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Key observation: The RHS depends only on the Clifford generators, corresponding to the non-diagonal entries of $\mathcal{R}^{\lambda}$.

If $b^{\mathcal{T}} = d(\mathcal{T})^{-1}(b)$ is a diagonal entry of $\mathcal{R}^{\lambda}$, then $b = d(\mathcal{T})(b^{\mathcal{T}})$ is a diagonal entry of $\mathcal{T}$, implying $\kappa_b(\mathcal{T}) = 0$.

Let $\mathcal{C}l^{\lambda}_{n-\ell}$ be the subalgebra of $\mathcal{C}l_n$ generated by $c_{b_1}, \ldots, c_{b_{n-\ell}}$, where $b_1 < \cdots < b_{n-\ell}$ are the non-diagonal entries of $\mathcal{R}^{\lambda}$.

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The superalgebra $\mathcal{C}l^{\lambda}_{n-\ell}$ has a unique right module E^{λ} of dimension 2^p, where $p = \lceil \frac{n-\ell}{2} \rceil$.

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The superalgebra $\mathcal{C}l^{\lambda}_{n-\ell}$ has a unique right module E^{λ} of dimension 2^p, where $p = \lceil \frac{n-\ell}{2} \rceil$. Let $\{\mathcal{F}^{\delta}\}$ be a basis of E^{λ}.

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The action of the generators t_a of $\mathbb{C}\mathfrak{S}_n^-$ in the basis is determined by the formulas of the Lemma.

Fusion procedure

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Take n variables $u_1, \dots, u_n$ and consider the rational function with values in $\mathbb{C}\mathfrak{S}_n$ defined by [Nazarov 1997],

$$\Phi(u_1, \dots, u_n) = \prod_{1 \leqslant a < b \leqslant n} \Big(1 - \frac{(a,b)}{u_a - u_b} + \frac{(a,b)\, c_a c_b}{u_a + v_b} \Big),$$

where the product is taken in the lexicographical order on the set of pairs (a, b).

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Let $\mathcal{U}$ be a standard barred λ-tableau. For every $a = 1, \ldots, n$ set $\kappa_a = \kappa_{\mathsf{a}}(\mathcal{U})$ if $\mathsf{a} = a$ or $\mathsf{a} = \bar{a}$ is the entry of $\mathcal{U}$.

Theorem. The consecutive evaluations are well-defined and we have

$$\Phi(u_1, \dots, u_n)\big|_{u_1=\kappa_1}\big|_{u_2=\kappa_2} \dots \big|_{u_n=\kappa_n} = \frac{n!}{g_\lambda}\, e_{\mathcal{U}}.$$

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Proof. This is essentially equivalent to the JM formula for $e_{\mathcal{U}}$.

Expect applications to representation theory of $\mathfrak{q}_N$ and the Yangian $\mathrm{Y}(\mathfrak{q}_N)$. Use the fact that the factors in $\Phi(u_1,\dots,u_n)$ are Nazarov R-matrices satisfying the Yang–Baxter equation.