Sydney University Mathematical Society Problem Competition 2008

This competition is open to undergraduates (including Honours students) at any Australian university or tertiary institution. Entrants may use any source of information except other people. The problems will also be posted on the web page **http://www.maths.usyd.edu.au/u/SUMS/**.

Entrants may submit solutions to as many problems as they wish. Prizes ($60 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Students from the University of Sydney are also eligible for the Norbert Quirk Prizes, based on the overall quality of their entry (one for each of 1st, 2nd and 3rd years). Extensions and generalizations of any problem are invited and are taken into account when assessing solutions. If two or more solutions to a problem are essentially equal, preference may be given to students in the earlier year of university; otherwise, prizes may be shared. If a problem receives no correct solutions, its prize-money will be redistributed among the other problems.

Entries must be received by **Friday, September 5, 2008**. They may be posted to Dr Anthony Henderson, School of Mathematics and Statistics, The University of Sydney, NSW 2006, or delivered in person to Room 805, Carslaw Building. Please mark your entry SUMS Problem Competition 2008, and include your name, university, student number, year of study, and postal address for the return of your entry and prizes.

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1. Imagine an analogue watch with the usual hour hand, minute hand, and second hand. At how many times each day are two of the hands pointing in exactly opposite directions?

2. A bee wants to fly on the real line from the point 0 to the point 1, visiting n flowers which are positioned at the points $\frac{1}{n+1}, \frac{2}{n+1}, \cdots, \frac{n}{n+1}$ (here n is some fixed positive integer). The bee chooses at random, with equal probabilities, one of the $n!$ possible orderings of the flowers. It flies from 0 to the first flower, from there to the second flower, and so on through all the flowers in the chosen order, before flying on to 1. What is the expected total distance it will fly?

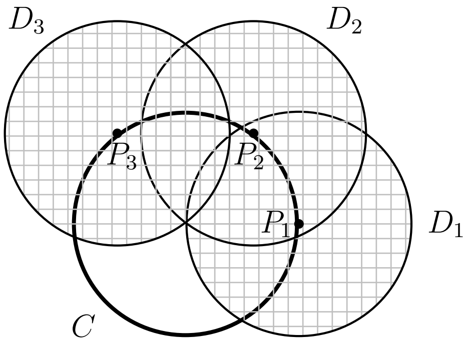
3. The sisters Alice and Bess want to practise their arithmetic, so their father invents the following game. He begins by choosing a composite number n_0 which is at least 6. Alice and Bess then take turns saying numbers $n_1, n_2, n_3, \cdots$ (with Alice saying n_1, Bess saying n_2, Alice saying n_3, and so on) in such a way that at each step the new number n_i is the sum of two integers ≥ 2 of which n_{i-1} is the product. The winner is the first player to say a prime number. For example, if $n_0 = 16$, then Alice can say either 8 or 10, because $8 = 4 + 4$ and $10 = 2 + 8$. Saying 10 would be a bad move, because Bess would then win by saying 7 (because $7 = 2 + 5$). So Alice should say 8, which forces Bess to say 6, allowing Alice to say 5 and win. Prove that there are infinitely many starting numbers n_0 for which Bess is guaranteed to win if she plays correctly, no matter what Alice does.

4. Let n be an odd integer ≥ 3, and let $x_1, x_2, \cdots, x_n$ be any real numbers, not necessarily positive. Prove that

$$(n-1)\max\{x_1^2, x_2^2, \cdots, x_n^2\} \ + \ (x_1 + x_2 + \cdots + x_n)^2 \ \geq \ x_1^2 + x_2^2 + \cdots + x_n^2.$$

5. Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be unit vectors in $\mathbb{R}^3$: that is, $|\mathbf{u}_i| = 1$ for all i, where $|\mathbf{w}|$ denotes the length of the vector $\mathbf{w}$. Assume that $|\mathbf{u}_1+\mathbf{u}_2+\cdots+\mathbf{u}_n| > n-2$, and that $\lambda_1\mathbf{u}_1+\lambda_2\mathbf{u}_2+\cdots+\lambda_n\mathbf{u}_n = 0$ for some nonnegative real numbers λ_i. Prove that $\lambda_i = 0$ for all i.

6. Fix a positive integer $n \geq 3$. Let $P_1, P_2, \cdots, P_n$ be points which lie on a circle C of radius 1, and let D_i denote the disc with centre P_i and radius 1. A possible picture when $n = 3$ is:



Find the maximum possible area of the union $D_1 \cup D_2 \cup \cdots \cup D_n$.

7. For real numbers a, b, c, d with $a \neq 0$, consider the equation $\overline{z} = az^3 + bz^2 + cz + d$, where the unknown z is a complex number and $\overline{z}$ denotes the conjugate of z. What are the minimum and maximum number of solutions this equation can have, for different choices of a, b, c, d?

8. A famous theorem in algebra says that any $n \times n$ integer matrix A can be written as a matrix product XDY, where X and Y are integer matrices with determinant ± 1, and D is a diagonal matrix with nonnegative integer diagonal entries $d_1, d_2, \cdots, d_n$ such that d_{i+1} is a multiple of d_i for all $1 \leq i \leq n-1$. The numbers $d_1, d_2, \cdots, d_n$ are uniquely determined and are called the *invariant factors* of A. Find the invariant factors of the matrix $A = (a_{ij})_{i,j=1}^n$, where $a_{ij} = i^j$.

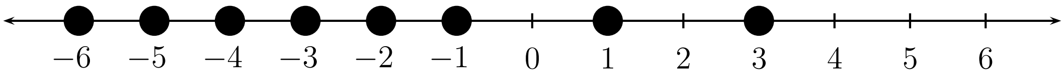
9. In the complex vector space $\mathbb{C}^2$ we define an inner product by

$$(z_1, z_2) \cdot (w_1, w_2) = z_1\overline{w_1} + z_2\overline{w_2}, \text{ for all } z_1, z_2, w_1, w_2 \in \mathbb{C}.$$

An element $(z_1, z_2) \in \mathbb{C}^2$ is a *unit vector* if $(z_1, z_2) \cdot (z_1, z_2) = 1$. Show that it is impossible to have five distinct unit vectors $(z_1^{(a)}, z_2^{(a)})$, $a = 1, 2, 3, 4, 5$, such that $|(z_1^{(a)}, z_2^{(a)}) \cdot (z_1^{(b)}, z_2^{(b)})|$ is the same for all pairs (a, b) with $a \neq b$.

10. Imagine placing infinitely many identical coins at integer points on the real line (at most one coin at each integer). Call such a placement *allowable* if, for all sufficiently large positive integers N, there is a coin at $-N$ but not at N. Thus every allowable placement has a contiguous block of coins on the left, and there is some integer a (the "first gap") which is minimal among those where there is no coin. Call an allowable placement *well-spaced* if there are no two coins at positions $b, b+1$ where $b > a$ (i.e. no adjacent coins to the right of the first gap). By a *move* from one allowable placement to another, we mean a move of a single coin two places to the right, i.e. removing a coin at i and replacing it at the previously empty position $i+2$ for some i.

For any integers m and n, define an allowable placement $P_{m,n}$ in which the coins are placed at the odd integers $\leq 2m-1$ and the even integers $\leq 2n$. Here is a picture of $P_{2,-1}$:



Let $f(m, n, w)$ be the number of well-spaced placements which can be obtained from $P_{m,n}$ by a sequence of exactly w moves (the intermediate placements do not have to be well-spaced). Prove that $f(m, n, w)$ is independent of m and n.