Sydney University Mathematical Society

Problems Competition 2000

This competition is open to all undergraduates at any Australian university or tertiary institution. Prizes ($40 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Entrants from the University of Sydney will also be eligible for the Norbert Quirk Prizes (one for each of 1st, 2nd and 3rd years).

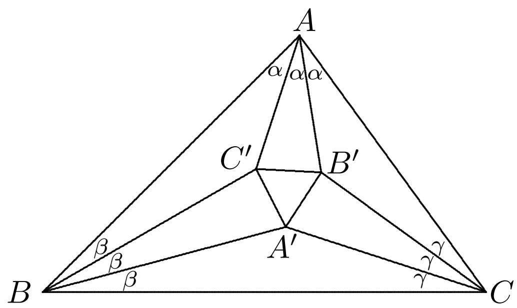
Contestants may use any source of information except other people. Solutions are to be received by 4.00 pm on Friday, October 6, 2000. They may be given to Dr. Donald Cartwright, Room 620, Carslaw Building, or posted to him at the School of Mathematics and Statistics, The University of Sydney, N.S.W. 2006. Entries must state name, university, student number, course and year, term address and telephone number, and be marked **2000 SUMS Competition**. The prizes will be awarded towards the end of the academic year.

The SUMS committee is grateful to all those who have provided problems. We are always keen to get more. Send any, with solutions, to Dr. Cartwright, at the above address.

Problems

(Extensions and generalizations of any problem are invited and are taken into account when assessing solutions.)

1. In a triangle ABC, the internal trisectors are drawn, intersecting in points A', B' and C' as shown.



Show that the triangle $\triangle A'B'C'$ is equilateral.

2. (a) Prove that there is no polyhedron with exactly seven edges. (b) For which natural numbers n does a polyhedron with n edges exist? (c) Find all the polyhedra with 11 edges.

3. We are interested in finding sets $\{a_1, \ldots, a_n\}$ of distinct positive integers with the property that $a_i + a_j$ is a perfect square for any $1 \le i < j \le n$. For example, $\{1, 24, 120\}$ is such a set, with $n = 3$. Can you find bigger sets?

4. Find all numbers n which can be written in one and only one way as the product $k_1 \times k_2 \times \cdots \times k_r$ of $r \geq 2$ positive integer factors $k_1 \geq k_2 \geq \cdots \geq k_r$ which add to n. For example, the number 6 can be written as the product $3 \times 2 \times 1$, and the factors 3, 2, and 1 add to 6. Up to order of the factors, this is the only way we can so write 6.

5. Is it possible to find four distinct real numbers $x_1, \ldots, x_4$ such that for any $i \neq j$, writing $x = x_i$ and $y = x_j$, the equation $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 1$ holds?

6. For $n = 1, 2, \ldots$, let p_n denote the number of partitions of n. For example, the partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$, and so $p_4 = 5$. Show that

$$p_1 + p_2 + \cdots + p_n < 2^n.$$

7. Let $n \geq 1$ be an integer. Recall that any permutation of $\{1, \ldots, n\}$ can be written as a product of cycles. For example, the permutation $3\,7\,4\,1\,6\,2\,5$ of $\{1, \ldots, 7\}$ equals the product $(1\,3\,4)(2\,7\,5\,6)$. Find the proportion of permutations of $\{1, \ldots, n\}$ which have a cycle of length greater than $n/2$. Show that this proportion is always at least $1/2$ and approaches $\ln 2$ as $n \to \infty$.

8. Suppose that we are given $2n - 1$ integers, with repetitions allowed. Show that it is possible to choose n of them whose sum is divisible by n.

9. Let V be a vector space of dimension 2^n, with a basis $\{e_S : S \subset \{1, \ldots, n\}\}$ whose elements are indexed by the subsets of $\{1, \ldots, n\}$. We can define two linear transformations X and Y of V by specifying their action on the basis vectors, namely, we set

$$X(e_S) = \sum_{\substack{S' : S \subset S' \\ |S'| = |S|+1}} e_{S'}$$

and

$$Y(e_s) = \sum_{\substack{S'' : S'' \subset S \\ |S''| = |S|-1}} e_{S''}.$$

Describe the linear transformation $XY - YX$.

10. Given fractions a/b and e/f, where $a, b, e, f > 0$ are integers, we define their median c/d to be the fraction $(a+e)/(b+f)$, reduced to "lowest terms"; we form the triple $(a/b, c/d, e/f)$ in this case. Form a new triple by deleting either a/b or e/f and replacing it by the median of the remaining two numbers. Given a rational number q, where $0 < q < 1$, prove that starting from the triple $(0/1, 1/2, 1/1)$, you will eventually end up with a unique triple with q as its middle term.