Sydney University Mathematical Society

Problems Competition 1998

This competition is open to all undergraduates at any Australian university or tertiary institution. Prizes ($40 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Entrants from the University of Sydney will also be eligible for the Norbert Quirk Prizes (one for each of 1st, 2nd and 3rd years).

Contestants may use any source of information except other people. Solutions are to be received by 4.00 pm on Friday, August 28, 1998. They may be given to Dr. Donald Cartwright, Room 620, Carslaw Building, or posted to him at the School of Mathematics and Statistics, The University of Sydney, N.S.W. 2006. Entries must state name, university, student number, course and year, term address and telephone number, and be marked **1998 SUMS Competition**. The prizes will be awarded towards the end of the academic year.

The SUMS committee is grateful to all those who have provided problems. We are always keen to get more. Send any, with solutions, to Dr. Cartwright, at the above address.

Problems

(Extensions and generalizations of any problem are invited and are taken into account when assessing solutions.)

1. We are given m letters "a" and n letters "b". Find the condition on m and n in order that one can form a palindrome using all these letters (a palindrome being a word, not necessarily an English one, which reads the same forwards and backwards). When this condition holds, how many distinct palindromes can you form? Generalize to m_1 letters "a_1", ..., m_r letters "a_r".

2. Under what conditions on the real coefficients c_0, c_1, c_2, c_3 does the quartic curve $y = x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$ meet some straight line in 4 distinct points?

3. A sphere can be expressed as a union of two overlapping "charts", one covering from the "north pole" down to a little below the "equator", and the other covering from the "south pole" to a little above the equator. How many charts do you need to cover a torus (a surface like a doughnut)? A chart here is a set homeomorphic to the unit disc $\{(x, y) : x^2 + y^2 < 1\}$.

4. You are given 3 boxes, and some marbles in the boxes. You were allowed to double the number of marbles in one of the boxes by taking some from one of the other boxes. By repeating this procedure, can you always get to the situation that one of the boxes is empty?

5. Find an example of a convergent series $\sum_{k=1}^{\infty} a_k$ of real numbers for which $\sum_{k=1}^{\infty} \sin a_k$ is divergent. Can you find such a series whose terms converge to 0 monotonically in absolute value?

6. Find

$$\lim_{n\to\infty} \min_t \left\{ \left(\frac{e^t + e^{2t} + e^{-3t}}{3} \right)^{n^2} e^{-tn} \right\}.$$

7. Let p be a prime. Show that the number of integers a in $\{0, 1, ..., p-1\}$ which are congruent modulo p to a number of the form $x^3 - 3x$ is 3 if $p = 3$ and $(2p \pm 1)/3$ if $p \equiv \pm 1$ mod 3.

8. Given a sequence $a_1, a_2, \ldots$ of positive numbers, show that there is a real number C such that

$$\frac{1}{a_n} \sum_{k=1}^{n} a_k \le C$$

for all $n \ge 1$ if and only if there is a number C' such that

$$a_n \sum_{k=n}^{\infty} \frac{1}{a_k} \le C'$$

for all n.

9. Let

$$0 = f_0 < f_1 < \cdots < f_k = n$$

and

$$0 = g_0 < g_1 < \cdots < g_m = n$$

be two strictly increasing sequences of integers. Let W_1 be the set of permutations of $\{1, \ldots, n\}$ fixing each set $\{f_i + 1, \ldots, f_{i+1}\}$, and similarly let W_2 be the set of permutations of $\{1, \ldots, n\}$ fixing each set $\{g_i + 1, \ldots, g_{i+1}\}$. Show that a permutation σ is a product of a permutation from W_1 and a permutation from W_2 if and only if for all i, j, with $1 \le i \le m$ and $1 \le j \le k$ and $f_{j-1} < g_i \le f_j$, we have $\{1, 2, \ldots, f_{j-1}\} \subset \sigma(\{1, 2 \ldots, g_i\}) \subset \{1, 2, \ldots, f_j\}$.

10. Consider a sequence $a_1, a_2, \ldots$ defined by the recurrence relation

$$a_n = a_{a_{n-1}} + a_{n-a_{n-1}} \quad \text{for } n \ge 3,$$

with the initial conditions $a_1 = a_2 = 1$. What is the value of a_n if n is a power of 2?