Besov Spaces Associated with Operators

Joint Seminar Talk

Anthony Wong

Macquarie University

October 1, 2014

Outline

- Classical Besov Spaces
- Besov norms associated with operators
  - Spaces of polynomial upper bounds on volume growth
  - Assumptions on operators
- Besov spaces associated with operators
- Properties of Besov spaces associated with operators
  - Embedding theorem
  - Norm equivalence
  - Fractional integrals
- Decomposition of Besov spaces associated with operators

MACQUARIE UNIVERSITY

Outline

- Classical Besov Spaces
- Besov norms associated with operators
  - Spaces of polynomial upper bounds on volume growth
  - Assumptions on operators
- Besov spaces associated with operators
- Properties of Besov spaces associated with operators
  - Embedding theorem
  - Norm equivalence
  - Fractional integrals
- Decomposition of Besov spaces associated with operators

Outline

- Classical Besov Spaces
- Besov norms associated with operators
  - Spaces of polynomial upper bounds on volume growth
  - Assumptions on operators
- Besov spaces associated with operators
- Properties of Besov spaces associated with operators
  - Embedding theorem
  - Norm equivalence
  - Fractional integrals
- Decomposition of Besov spaces associated with operators

Classical Besov Spaces

The theory of Besov spaces has been an active area of research in the last few decades because of its important role in the study of approximation of functions and regularity of solutions to partial differential equations.

Let φ be a function which satisfies the following conditions:

$$\varphi \in C_0^\infty(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \varphi(x)\,dx = 0,$$

and the standard Tauberian condition, that is

$$\forall \xi \neq 0, \exists t = t_\xi > 0 \quad \text{such that} \quad \hat{\varphi}(t\xi) \neq 0.$$

The theory of Besov spaces has been an active area of research in the last few decades because of its important role in the study of approximation of functions and regularity of solutions to partial differential equations.

Let φ be a function which satisfies the following conditions:

$$\varphi \in C_0^\infty(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \varphi(x)\, dx = 0,$$

and the standard Tauberian condition, that is

$$\forall \xi \neq 0, \exists t = t_\xi > 0 \quad \text{such that} \quad \hat{\varphi}(t\xi) \neq 0.$$

Classical Besov Spaces

The theory of Besov spaces has been an active area of research in the last few decades because of its important role in the study of approximation of functions and regularity of solutions to partial differential equations.

Let φ be a function which satisfies the following conditions:

$$\varphi \in C_0^\infty(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \varphi(x)\,dx = 0,$$

and the standard Tauberian condition, that is

$$\forall \xi \neq 0, \exists t = t_\xi > 0 \quad \text{such that} \quad \hat{\varphi}(t\xi) \neq 0.$$

MACQUARIE UNIVERSITY

Classical Besov Spaces

We use $\varphi_t, t > 0$, to denote the dilation of φ:

$$\varphi_t(x) = t^{-n}\varphi(x/t), \quad x \in \mathbb{R}^n.$$

For $-1 < \alpha < 1$, $1 \le p, q \le \infty$, the classical (homogeneous) Besov space $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ on the Euclidean space $\mathbb{R}^n$ can be defined as follows:

$$\dot{B}^{\alpha}_{p,q} = \left\{ f \in \mathcal{S}' : \left[\int_0^\infty (t^{-\alpha}\|\varphi_t * (f)\|_p)^q \frac{dt}{t} \right]^{1/q} = \|f\|_{\dot{B}^{\alpha}_{p,q}} < \infty \right\}.$$

where $\mathcal{S}'$ is the space of tempered distributions.

MACQUARIE UNIVERSITY

Classical Besov Spaces

We use $\varphi_t, t > 0$, to denote the dilation of φ:

$$\varphi_t(x) = t^{-n}\varphi(x/t), \quad x \in \mathbb{R}^n.$$

For $-1 < \alpha < 1$, $1 \leq p, q \leq \infty$, the classical (homogeneous) Besov space $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ on the Euclidean space $\mathbb{R}^n$ can be defined as follows:

$$\dot{B}^{\alpha}_{p,q} = \left\{ f \in \mathcal{S}' : \left[\int_0^{\infty} (t^{-\alpha} \|\varphi_t * (f)\|_p)^q \frac{dt}{t} \right]^{1/q} = \|f\|_{\dot{B}^{\alpha}_{p,q}} < \infty \right\}.$$

where $\mathcal{S}'$ is the space of tempered distributions.

MACQUARIE UNIVERSITY

Classical Besov Spaces

We use $\varphi_t, t > 0$, to denote the dilation of φ:

$$\varphi_t(x) = t^{-n}\varphi(x/t), \quad x \in \mathbb{R}^n.$$

For $-1 < \alpha < 1$, $1 \leq p, q \leq \infty$, the classical (homogeneous) Besov space $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ on the Euclidean space $\mathbb{R}^n$ can be defined as follows:

$$\dot{B}^{\alpha}_{p,q} = \left\{ f \in \mathcal{S}' : \left[\int_0^{\infty} (t^{-\alpha} \|\varphi_t * (f)\|_p)^q \frac{dt}{t} \right]^{1/q} = \|f\|_{\dot{B}^{\alpha}_{p,q}} < \infty \right\}.$$

where $\mathcal{S}'$ is the space of tempered distributions.

Besov spaces associated with operators

Recent work in the paper:

H.-Q. Bui, X.T. Duong, L.X. Yan, Calderón Reproducing Formulas and New Besov Spaces Associated with Operators, *Advances in Mathematics*, **229** (2012), 2449–2502.

defined Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Poisson kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth.

When L is the Laplace operator $-\Delta$ or its square root $\sqrt{-\Delta}$ acting on the Euclidean space $\mathbb{R}^n$, this class of Besov spaces associated with the operator L are equivalent to the classical Besov spaces.

MACQUARIE UNIVERSITY

Recent work in the paper:

H.-Q. Bui, X.T. Duong, L.X. Yan, Calderón Reproducing Formulas and New Besov Spaces Associated with Operators, *Advances in Mathematics*, **229** (2012), 2449–2502.

defined Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Poisson kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth.

When L is the Laplace operator $-\Delta$ or its square root $\sqrt{-\Delta}$ acting on the Euclidean space $\mathbb{R}^n$, this class of Besov spaces associated with the operator L are equivalent to the classical Besov spaces.

Recent work in the paper:

H.-Q. Bui, X.T. Duong, L.X. Yan, Calderón Reproducing Formulas and New Besov Spaces Associated with Operators, *Advances in Mathematics*, **229** (2012), 2449–2502.

defined Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Poisson kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth.

When L is the Laplace operator $-\Delta$ or its square root $\sqrt{-\Delta}$ acting on the Euclidean space $\mathbb{R}^n$, this class of Besov spaces associated with the operator L are equivalent to the classical Besov spaces.

Besov spaces associated with operators

A. Wong, Besov Spaces Associated with Operators, *Commun. Math. Anal.*, **16** (2014), 89–104.

In the above paper we extend certain results from the case when the underlying space has only one dimension,

that is,

$$\mu(B(x,r)) \leq Cr^n, \quad r > 0$$

to a more general setting when the underlying space can have different dimensions at 0 and ∞,

that is,

$$\mu(B(x,r)) \leq \begin{cases} Cr^n, & 0 < r \leq 1 \\ Cr^N, & 1 < r < \infty \end{cases}$$

An example of this case is in Lie groups of polynomial growth.

Besov spaces associated with operators

A. Wong, Besov Spaces Associated with Operators, *Commun. Math. Anal.*, **16** (2014), 89–104.

In the above paper we extend certain results from the case when the underlying space has only one dimension,

that is,

$$\mu(B(x,r)) \leq Cr^n, \quad r > 0$$

to a more general setting when the underlying space can have different dimensions at 0 and ∞,

that is,

$$\mu(B(x,r)) \leq \begin{cases} Cr^n, & 0 < r \leq 1 \\ Cr^N, & 1 < r < \infty \end{cases}$$

An example of this case is in Lie groups of polynomial growth.

Besov spaces associated with operators

A. Wong, Besov Spaces Associated with Operators, *Commun. Math. Anal.*, **16** (2014), 89–104.

In the above paper we extend certain results from the case when the underlying space has only one dimension,

that is,

$$\mu(B(x, r)) \leq Cr^n, \quad r > 0$$

to a more general setting when the underlying space can have different dimensions at 0 and ∞,

that is,

$$\mu(B(x, r)) \leq \begin{cases} Cr^n, & 0 < r \leq 1 \\ Cr^N, & 1 < r < \infty \end{cases}$$

An example of this case is in Lie groups of polynomial growth.

Assume $\mathcal{X}$ is a quasi-metric measure space satisfying

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(iii) There exists a constant $C \in [1, \infty)$ such that for all x, y and $z \in \mathcal{X}$, $d(x, y) \le C(d(x, z) + d(z, y))$;

(iv) For some $n, N > 0$, and $C > 0$,

$$\mu(B(x,r)) \le \begin{cases} Cr^n, & 0 < r \le 1 \\ Cr^N, & 1 < r < \infty \end{cases}$$

for all balls B. Here n is the local dimension and N is the global dimension or the dimension at infinity.

Assume $\mathcal{X}$ is a quasi-metric measure space satisfying

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(iii) There exists a constant $C \in [1, \infty)$ such that for all x, y and $z \in \mathcal{X}$, $d(x, y) \leq C(d(x, z) + d(z, y))$;

(iv) For some $n, N > 0$, and $C > 0$,

$$\mu(B(x, r)) \leq \begin{cases} Cr^n, & 0 < r \leq 1 \\ Cr^N, & 1 < r < \infty \end{cases}$$

for all balls B. Here n is the local dimension and N is the global dimension or the dimension at infinity.

MACQUARIE UNIVERSITY

Assume $\mathcal{X}$ is a quasi-metric measure space satisfying

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in \mathcal{X}$;

(iii) There exists a constant $C \in [1, \infty)$ such that for all x, y and $z \in \mathcal{X}$, $d(x,y) \leq C(d(x,z) + d(z,y))$;

(iv) For some $n, N > 0$, and $C > 0$,

$$\mu(B(x,r)) \leq \begin{cases} Cr^n, & 0 < r \leq 1 \\ Cr^N, & 1 < r < \infty \end{cases}$$

for all balls B. Here n is the local dimension and N is the global dimension or the dimension at infinity.

Assume L is densely-defined on $L^2(\mathcal{X})$ and satisfies

(S) L generates a holomorphic semigroup e^{-zL} for $z = t + is$ with $t > 0$ and $|\arg z| < \rho$ for some $\rho > 0$,

(K) the heat kernel of L satisfies bounds of Gaussian type, i.e. the kernel $p_t(x, y)$ of e^{-tL} satisfies

$$|p_t(x,y)| \leq \begin{cases} \dfrac{C}{t^{n/2}} e^{-\alpha d(x,y)^2/t}, & 0 < t \leq 1 \\ \dfrac{C}{t^{N/2}} e^{-\alpha d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

for some $C > 0$ and for all $x, y \in \mathcal{X}$.

Assume L is densely-defined on $L^2(\mathcal{X})$ and satisfies

(S) L generates a holomorphic semigroup e^{-zL} for $z = t + is$ with $t > 0$ and $|\arg z| < \rho$ for some $\rho > 0$,

(K) the heat kernel of L satisfies bounds of Gaussian type, i.e. the kernel $p_t(x, y)$ of e^{-tL} satisfies

$$|p_t(x,y)| \leq \begin{cases} \dfrac{C}{t^{n/2}} e^{-\alpha d(x,y)^2/t}, & 0 < t \leq 1 \\ \dfrac{C}{t^{N/2}} e^{-\alpha d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

for some $C > 0$ and for all $x, y \in \mathcal{X}$.

Assume L is densely-defined on $L^2(\mathcal{X})$ and satisfies

- (S) L generates a holomorphic semigroup e^{-zL} for $z = t + is$ with $t > 0$ and $|\arg z| < \rho$ for some $\rho > 0$,
- (K) the heat kernel of L satisfies bounds of Gaussian type, i.e. the kernel $p_t(x, y)$ of e^{-tL} satisfies

$$|p_t(x,y)| \leq \begin{cases} \dfrac{C}{t^{n/2}} e^{-\alpha d(x,y)^2/t}, & 0 < t \leq 1 \\ \dfrac{C}{t^{N/2}} e^{-\alpha d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

for some $C > 0$ and for all $x, y \in \mathcal{X}$.

Besov norms associated with operators

For $k = 1, 2, \ldots$, let $p_{k,t}(x, y)$ denote the kernel of the operator $t^k L^k e^{-tL}$. Suppose L satisfies (S) and (K).

Then $p_{k,t}(x, y)$ satisfies the size estimate (DK), i.e. for every $k \in \mathbb{N}$, there is a constant c_k satisfying

$$|p_{k,t}(x, y)| \leq \begin{cases} \frac{c_k}{t^{n/2}} e^{-\alpha_k d(x,y)^2/t}, & 0 < t \leq 1 \\ \frac{c_k}{t^{N/2}} e^{-\alpha_k d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

for all $x, y \in \mathcal{X}$.

Besov norms associated with operators

For $k = 1, 2, \ldots,$ let $p_{k,t}(x, y)$ denote the kernel of the operator $t^k L^k e^{-tL}$. Suppose L satisfies (S) and (K).

Then $p_{k,t}(x, y)$ satisfies the size estimate (DK), i.e. for every $k \in \mathbb{N}$, there is a constant c_k satisfying

$$|p_{k,t}(x, y)| \le \begin{cases} \frac{c_k}{t^{n/2}} e^{-\alpha_k d(x,y)^2/t}, & 0 < t \le 1 \\ \frac{c_k}{t^{N/2}} e^{-\alpha_k d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

for all $x, y \in \mathcal{X}$.

Besov norms associated with operators

For $k = 1, 2, \ldots,$ let $p_{k,t}(x, y)$ denote the kernel of the operator $t^k L^k e^{-tL}$. Suppose L satisfies (S) and (K).

Then $p_{k,t}(x, y)$ satisfies the size estimate (DK), i.e. for every $k \in \mathbb{N}$, there is a constant c_k satisfying

$$|p_{k,t}(x,y)| \leq \begin{cases} \dfrac{c_k}{t^{n/2}} e^{-\alpha_k d(x,y)^2/t}, & 0 < t \leq 1 \\ \dfrac{c_k}{t^{N/2}} e^{-\alpha_k d(x,y)^2/t}, & 1 < t < \infty \end{cases}$$

for all $x, y \in \mathcal{X}$.

Besov norms associated with operators

Assume L satisfies (S) and (K). Let $k_t(x, y) = p_{1,t}(x, y)$ be the kernel of $\Psi_t(L) = tLe^{-tL}$. Let f be a complex valued measurable function on $\mathcal{X}$ satisfying the following growth condition (G):

$$\int_{\mathcal{X}} |f(x)|e^{-\alpha d(x,y_0)^2} \, d\mu(x) < \infty$$

for some $y_0 \in \mathcal{X}$.

Then we have that

$$\Psi_t(L)f(x) = \int_{\mathcal{X}} k_t(x, y)f(y) \, d\mu(y)$$

is defined for all $x \in \mathcal{X}$.

Besov norms associated with operators

Assume L satisfies (S) and (K). Let $k_t(x,y) = p_{1,t}(x,y)$ be the kernel of $\Psi_t(L) = tLe^{-tL}$. Let f be a complex valued measurable function on $\mathcal{X}$ satisfying the following growth condition (G):

$$\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,y_0)^2}\, d\mu(x) < \infty$$

for some $y_0 \in \mathcal{X}$.

Then we have that

$$\Psi_t(L)f(x) = \int_{\mathcal{X}} k_t(x,y) f(y)\, d\mu(y)$$

is defined for all $x \in \mathcal{X}$.

Besov norms associated with operators

Assume L satisfies (S) and (K). Let $k_t(x,y) = p_{1,t}(x,y)$ be the kernel of $\Psi_t(L) = tLe^{-tL}$. Let f be a complex valued measurable function on $\mathcal{X}$ satisfying the following growth condition (G):

$$\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,y_0)^2} \, d\mu(x) < \infty$$

for some $y_0 \in \mathcal{X}$.

Then we have that

$$\Psi_t(L) f(x) = \int_{\mathcal{X}} k_t(x,y) f(y) \, d\mu(y)$$

is defined for all $x \in \mathcal{X}$.

MACQUARIE UNIVERSITY

Besov norms associated with operators

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$.

For any f satisfying (G), we define its $\dot{B}^{\alpha,L}_{p,q}$-norm by

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \left\{ \int_0^\infty (t^{-\alpha} \|\Psi_t(L) f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \sup_{t>0} t^{-\alpha} \|\Psi_t(L) f\|_p$$

for $q = \infty$, whenever these are finite.

MACQUARIE UNIVERSITY

Besov norms associated with operators

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$.

For any f satisfying (G), we define its $\dot{B}^{\alpha,L}_{p,q}$-norm by

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \left\{ \int_0^\infty (t^{-\alpha} \|\Psi_t(L) f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \sup_{t>0} t^{-\alpha} \|\Psi_t(L) f\|_p$$

for $q = \infty$, whenever these are finite.

MACQUARIE UNIVERSITY

Besov norms associated with operators

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$.

For any f satisfying (G), we define its $\dot{B}^{\alpha,L}_{p,q}$-norm by

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \sup_{t>0} t^{-\alpha}\|\Psi_t(L)f\|_p$$

for $q = \infty$, whenever these are finite.

Besov norms associated with operators

There exists functions with finite Besov norm but not necessarily smooth. The following result gives an upper bound estimate of the Besov norm of the heat kernels.

For any $k \in \mathbb{N}$, we denote $\Psi_{k,t}(L) = t^k L^k e^{-tL}$ to be the operator whose kernel is $p_{k,t}$; so $\Psi_{1,t}(L) = \Psi_t(L)$.

Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Suppose that f satisfies (S) and (K). Then for $k \in \mathbb{N}$ and $z \in \mathcal{X}$,

$$\|p_{k,s}(\cdot, z)\|_{\dot{B}^{\alpha,L}_{p,q}} \leq \begin{cases} C_n s^{-\alpha - n/2p'}, & 0 < s \leq 1 \\ C_N s^{-\alpha - N/2p'}, & 1 < s < \infty \end{cases}$$

where $C_n > 0$ depends on α, n, k, p and q, and $C_N > 0$ depends on α, N, k, p and q.

Besov norms associated with operators

There exists functions with finite Besov norm but not necessarily smooth. The following result gives an upper bound estimate of the Besov norm of the heat kernels.

For any $k \in \mathbb{N}$, we denote $\Psi_{k,t}(L) = t^k L^k e^{-tL}$ to be the operator whose kernel is $p_{k,t}$; so $\Psi_{1,t}(L) = \Psi_t(L)$.

Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. Suppose that f satisfies (S) and (K). Then for $k \in \mathbb{N}$ and $z \in \mathcal{X}$,

$$\|p_{k,s}(\cdot, z)\|_{\dot{B}^{\alpha,L}_{p,q}} \le \begin{cases} C_n s^{-\alpha - n/2p'}, & 0 < s \le 1 \\ C_N s^{-\alpha - N/2p'}, & 1 < s < \infty \end{cases}$$

where $C_n > 0$ depends on α, n, k, p and q, and $C_N > 0$ depends on α, N, k, p and q.

MACQUARIE UNIVERSITY

Besov norms associated with operators

There exists functions with finite Besov norm but not necessarily smooth. The following result gives an upper bound estimate of the Besov norm of the heat kernels.

For any $k \in \mathbb{N}$, we denote $\Psi_{k,t}(L) = t^k L^k e^{-tL}$ to be the operator whose kernel is $p_{k,t}$; so $\Psi_{1,t}(L) = \Psi_t(L)$.

Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Suppose that f satisfies (S) and (K). Then for $k \in \mathbb{N}$ and $z \in \mathcal{X}$,

$$\|p_{k,s}(\cdot, z)\|_{\dot{B}^{\alpha,L}_{p,q}} \leq \begin{cases} C_n s^{-\alpha - n/2p'}, & 0 < s \leq 1 \\ C_N s^{-\alpha - N/2p'}, & 1 < s < \infty \end{cases}$$

where $C_n > 0$ depends on α, n, k, p and q, and $C_N > 0$ depends on α, N, k, p and q.

Besov spaces associated with operators

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \le p, q \le \infty$. A function f is in the space of test functions $\mathcal{M}_{p,q}^{\alpha,L}$ if $f = Lg$ for some g, and the following are satisfied:

(i) $\|f\|_{\dot{B}_{p,q}^{\alpha,L}} < \infty$;

(ii) There is a $C > 0$ such that

$$|f(x)| + |g(x)| \le Ce^{-\alpha d(x,x_0)^2}$$

for some $x_0 \in \mathcal{X}$, and for every $x \in \mathcal{X}$.

For $q = \infty$, we assume, in addition, that

$$\|t^{-\alpha}\Psi_t(L)f\|_p \to 0 \text{ as } t \to 0 \text{ or } t \to \infty,$$

and when $p = \infty$, we assume that

$$e^{-sL}f \to f \text{ in } \dot{B}_{\infty,q}^{\alpha,L} \text{ as } s \to 0.$$

Besov spaces associated with operators

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. A function f is in the space of test functions $\mathcal{M}_{p,q}^{\alpha,L}$ if $f = Lg$ for some g, and the following are satisfied:

(i) $\|f\|_{\dot{B}_{p,q}^{\alpha,L}} < \infty$;

(ii) There is a $C > 0$ such that

$$|f(x)| + |g(x)| \leq Ce^{-\alpha d(x,x_0)^2}$$

for some $x_0 \in \mathcal{X}$, and for every $x \in \mathcal{X}$.

For $q = \infty$, we assume, in addition, that

$$\|t^{-\alpha}\Psi_t(L)f\|_p \to 0 \text{ as } t \to 0 \text{ or } t \to \infty,$$

and when $p = \infty$, we assume that

$$e^{-sL}f \to f \text{ in } \dot{B}_{\infty,q}^{\alpha,L} \text{ as } s \to 0.$$

MACQUARIE UNIVERSITY

Besov spaces associated with operators

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. A function f is in the space of test functions $\mathcal{M}_{p,q}^{\alpha,L}$ if $f = Lg$ for some g, and the following are satisfied:

(i) $\|f\|_{\dot{B}_{p,q}^{\alpha,L}} < \infty$;

(ii) There is a $C > 0$ such that

$$|f(x)| + |g(x)| \leq Ce^{-\alpha d(x,x_0)^2}$$

for some $x_0 \in \mathcal{X}$, and for every $x \in \mathcal{X}$.

For $q = \infty$, we assume, in addition, that

$$\|t^{-\alpha}\Psi_t(L)f\|_p \to 0 \text{ as } t \to 0 \text{ or } t \to \infty,$$

and when $p = \infty$, we assume that

$$e^{-sL}f \to f \text{ in } \dot{B}_{\infty,q}^{\alpha,L} \text{ as } s \to 0.$$

Besov spaces associated with operators

We define the Besov space $\dot{B}^{\alpha,L}_{p,q}$ associated to an operator L by

$$\dot{B}^{\alpha,L}_{p,q} = \left\{ f \in (\mathcal{M}^{-\alpha,L^*}_{p',q'})' : \|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty \right\}.$$

where $\mathcal{M}^{\alpha,L}_{p,q}$ is a certain space of test functions,

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \left[\int_0^\infty (t^{-\alpha} \|\Psi_t(L)(f)\|_p)^q \frac{dt}{t} \right]^{1/q},$$

where

$$\Psi_t(L)f(x) = \int_{\mathcal{X}} k_t(x,y) f(y) \, d\mu(y)$$

where $k_t(x,y)$ is the kernel of $\Psi_t(L) = tLe^{-tL}$.

MACQUARIE UNIVERSITY

Besov spaces associated with operators

We define the Besov space $\dot{B}^{\alpha,L}_{p,q}$ associated to an operator L by

$$\dot{B}^{\alpha,L}_{p,q} = \left\{ f \in \left(\mathcal{M}^{-\alpha,L^*}_{p',q'}\right)' : \|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty \right\}.$$

where $\mathcal{M}^{\alpha,L}_{p,q}$ is a certain space of test functions,

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \left[\int_0^\infty (t^{-\alpha} \|\Psi_t(L)(f)\|_p)^q \frac{dt}{t} \right]^{1/q},$$

where

$$\Psi_t(L)f(x) = \int_{\mathcal{X}} k_t(x,y) f(y)\, d\mu(y)$$

where $k_t(x,y)$ is the kernel of $\Psi_t(L) = tLe^{-tL}$.

Besov spaces associated with operators

We define the Besov space $\dot{B}^{\alpha,L}_{p,q}$ associated to an operator L by

$$\dot{B}^{\alpha,L}_{p,q} = \left\{ f \in \left(\mathcal{M}^{-\alpha,L^*}_{p',q'}\right)' : \|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty \right\}.$$

where $\mathcal{M}^{\alpha,L}_{p,q}$ is a certain space of test functions,

$$\|f\|_{\dot{B}^{\alpha,L}_{p,q}} = \left[\int_0^\infty (t^{-\alpha} \|\Psi_t(L)(f)\|_p)^q \frac{dt}{t} \right]^{1/q},$$

where

$$\Psi_t(L)f(x) = \int_{\mathcal{X}} k_t(x,y) f(y)\, d\mu(y)$$

where $k_t(x,y)$ is the kernel of $\Psi_t(L) = tLe^{-tL}$.

The following Calderón reproducing formulas play an important role in the theory of function spaces and are of independent interest. They are used to study properties of the Besov spaces.

Suppose L is a densely-defined operator in $L^2(\mathcal{X})$ and satisfies (S) and (K). Assume that $f \in L^p(\mathcal{X})$, $1 < p < \infty$. Then we have

$$f(x) = \frac{1}{(k-1)!} \int_0^\infty t^k L^k e^{-tL} f(x) \frac{dt}{t}, \quad k = 1, 2, \ldots,$$

where the integral converges strongly in $L^p(\mathcal{X})$.

MACQUARIE UNIVERSITY

The following Calderón reproducing formulas play an important role in the theory of function spaces and are of independent interest. They are used to study properties of the Besov spaces.

Suppose L is a densely-defined operator in $L^2(\mathcal{X})$ and satisfies (S) and (K). Assume that $f \in L^p(\mathcal{X})$, $1 < p < \infty$. Then we have

$$f(x) = \frac{1}{(k-1)!} \int_0^\infty t^k L^k e^{-tL} f(x) \frac{dt}{t}, \quad k = 1, 2, \ldots,$$

where the integral converges strongly in $L^p(\mathcal{X})$.

The following Calderón reproducing formulas play an important role in the theory of function spaces and are of independent interest. They are used to study properties of the Besov spaces.

Suppose L is a densely-defined operator in $L^2(\mathcal{X})$ and satisfies (S) and (K). Assume that $f \in L^p(\mathcal{X})$, $1 < p < \infty$. Then we have

$$f(x) = \frac{1}{(k-1)!} \int_0^\infty t^k L^k e^{-tL} f(x) \frac{dt}{t}, \quad k = 1, 2, \ldots,$$

where the integral converges strongly in $L^p(\mathcal{X})$.

Suppose L satisfies (S) and (K), and L^* its adjoint operator. Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Let p' and q' be the conjugate exponents of p and q respectively.

Then for $\Psi_t(L) = tLe^{-tL}$, we have

$$(f, \phi) = 4 \int_0^\infty \int_{\mathcal{X}} \Psi_t(L) f(x) \Psi_t(L^*) \phi(x) \, d\mu(x) \, \frac{dt}{t}$$

for every $f \in \dot{B}^{\alpha,L}_{p,q}$ and $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

Suppose L satisfies (S) and (K), and L^* its adjoint operator. Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Let p' and q' be the conjugate exponents of p and q respectively.

Then for $\Psi_t(L) = tLe^{-tL}$, we have

$$(f, \phi) = 4 \int_0^\infty \int_{\mathcal{X}} \Psi_t(L)f(x)\Psi_t(L^*)\phi(x)\, d\mu(x)\, \frac{dt}{t}$$

for every $f \in \dot{B}^{\alpha,L}_{p,q}$ and $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

Suppose L satisfies (S) and (K), and L^* its adjoint operator. Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Let p' and q' be the conjugate exponents of p and q respectively.

Then for $\Psi_t(L) = tLe^{-tL}$, we have

$$(f, \phi) = 4 \int_0^\infty \int_{\mathcal{X}} \Psi_t(L) f(x) \Psi_t(L^*) \phi(x) \, d\mu(x) \, \frac{dt}{t}$$

for every $f \in \dot{B}^{\alpha,L}_{p,q}$ and $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

Suppose L satisfies (S) and (K), and L^* its adjoint operator. Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Let p' and q' be the conjugate exponents of p and q respectively. Also suppose $\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,x_0)^2}\, d\mu(x) < \infty$ for some $x_0 \in \mathcal{X}$, and $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty$.

Then for $\Psi_t(L) = tLe^{-tL}$, we have

$$(f, \phi) = \int_{\mathcal{X}} f(x)\phi(x)\, d\mu(x) = 4\int_0^\infty \int_{\mathcal{X}} \Psi_t(L)f(x)\Psi_t(L^*)\phi(x)\, d\mu(x)\, \frac{dt}{t}$$

for every $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

MACQUARIE UNIVERSITY

Suppose L satisfies (S) and (K), and L^* its adjoint operator. Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Let p' and q' be the conjugate exponents of p and q respectively. Also suppose $\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,x_0)^2} \, d\mu(x) < \infty$ for some $x_0 \in \mathcal{X}$, and $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty$.

Then for $\Psi_t(L) = tLe^{-tL}$, we have

$$(f, \phi) = \int_{\mathcal{X}} f(x)\phi(x) \, d\mu(x) = 4 \int_0^\infty \int_{\mathcal{X}} \Psi_t(L) f(x) \Psi_t(L^*)\phi(x) \, d\mu(x) \, \frac{dt}{t}$$

for every $\phi \in \mathcal{M}^{-\alpha,L^*}_{p',q'}$.

Suppose L satisfies (S) and (K), and L^* its adjoint operator. Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Let p' and q' be the conjugate exponents of p and q respectively. Also suppose $\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,x_0)^2} \, d\mu(x) < \infty$ for some $x_0 \in \mathcal{X}$, and $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty$.

Then for $\Psi_t(L) = tLe^{-tL}$, we have

$$(f, \phi) = \int_{\mathcal{X}} f(x)\phi(x) \, d\mu(x) = 4 \int_0^\infty \int_{\mathcal{X}} \Psi_t(L) f(x) \Psi_t(L^*)\phi(x) \, d\mu(x) \, \frac{dt}{t}$$

for every $\phi \in \mathcal{M}^{-\alpha, L^*}_{p', q'}$.

Suppose that L satisfies (S) and (K). Let $-1 < \alpha < 1$, $1 \le p, q \le \infty$ and $1 \le q_1 \le q_2 \le \infty$. Then the following statements are true:

(i) $\dot{B}^{\alpha,L}_{p,q} \subseteq (\mathcal{M}^{-\alpha,L^*}_{p',q'})'$ (continuous embedding).

(ii) $\dot{B}^{\alpha,L}_{p,q_1} \subseteq \dot{B}^{\alpha,L}_{p,q_2}$.

(iii) $\dot{B}^{\alpha,L}_{p,q}$ is complete.

(iv) If $1 \le p_1 \le p_2 \le \infty$, $-1 < \alpha_2 \le \alpha_1 < 1$ and $\alpha_1 - \frac{\min(n,N)}{2p_1} = \alpha_2 - \frac{\min(n,N)}{2p_2}$, then

$$\dot{B}^{\alpha_1,L}_{p_1,q} \subseteq \dot{B}^{\alpha_2,L}_{p_2,q},$$

Suppose that L satisfies (S) and (K). Let $-1 < \alpha < 1$, $1 \le p, q \le \infty$ and $1 \le q_1 \le q_2 \le \infty$. Then the following statements are true:

(i) $\dot{B}^{\alpha,L}_{p,q} \subseteq (\mathcal{M}^{-\alpha,L^*}_{p',q'})'$ (continuous embedding).

(ii) $\dot{B}^{\alpha,L}_{p,q_1} \subseteq \dot{B}^{\alpha,L}_{p,q_2}$.

(iii) $\dot{B}^{\alpha,L}_{p,q}$ is complete.

(iv) If $1 \le p_1 \le p_2 \le \infty$, $-1 < \alpha_2 \le \alpha_1 < 1$ and $\alpha_1 - \frac{\min(n,N)}{2p_1} = \alpha_2 - \frac{\min(n,N)}{2p_2}$, then

$$\dot{B}^{\alpha_1,L}_{p_1,q} \subseteq \dot{B}^{\alpha_2,L}_{p_2,q},$$

MACQUARIE UNIVERSITY

Suppose that L satisfies (S) and (K). Let $-1 < \alpha < 1$, $1 \leq p, q \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Then the following statements are true:

(i) $\dot{B}^{\alpha,L}_{p,q} \subseteq (\mathcal{M}^{-\alpha,L^*}_{p',q'})'$ (continuous embedding).

(ii) $\dot{B}^{\alpha,L}_{p,q_1} \subseteq \dot{B}^{\alpha,L}_{p,q_2}$.

(iii) $\dot{B}^{\alpha,L}_{p,q}$ is complete.

(iv) If $1 \leq p_1 \leq p_2 \leq \infty$, $-1 < \alpha_2 \leq \alpha_1 < 1$ and $\alpha_1 - \frac{\min(n,N)}{2p_1} = \alpha_2 - \frac{\min(n,N)}{2p_2}$, then

$$\dot{B}^{\alpha_1,L}_{p_1,q} \subseteq \dot{B}^{\alpha_2,L}_{p_2,q},$$

MACQUARIE UNIVERSITY

The following result shows that $\dot{B}^{\alpha,L}_{p,q}$ contains functions satisfying some growth condition.

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Suppose that f satisfies the growth condition (G):

$$\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,y_0)^2} \, d\mu(x) < \infty$$

and $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty$. Then $f \in \dot{B}^{\alpha,L}_{p,q}$.

The following result shows that $\dot{B}^{\alpha,L}_{p,q}$ contains functions satisfying some growth condition.

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Suppose that f satisfies the growth condition (G):

$$\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,y_0)^2} \, d\mu(x) < \infty$$

and $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty$. Then $f \in \dot{B}^{\alpha,L}_{p,q}$.

The following result shows that $\dot{B}^{\alpha,L}_{p,q}$ contains functions satisfying some growth condition.

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. Suppose that f satisfies the growth condition (G):

$$\int_{\mathcal{X}} |f(x)| e^{-\alpha d(x,y_0)^2} \, d\mu(x) < \infty$$

and $\|f\|_{\dot{B}^{\alpha,L}_{p,q}} < \infty$. Then $f \in \dot{B}^{\alpha,L}_{p,q}$.

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p \leq \infty$. The following three statements are equivalent for $f \in (\mathcal{M}_{p',q'}^{-\alpha,L^*})'$.

(a) $f \in \dot{B}_{p,q}^{\alpha,L}$

(b) f satisfies $\left\{ \sum_{j=-\infty}^{\infty} (2^{j\alpha} \|\Psi_{2^{-j}}(L)f\|_p)^q \right\}^{1/q} < \infty$

(c) f satisfies $\left\{ \sum_{j=-\infty}^{\infty} (2^{j\alpha} \|\Delta_j(L)f\|_p)^q \right\}^{1/q} < \infty$

where $\Delta_j(L)f = e^{-2^{-j}L}f - e^{-2^{-j-1}L}f$.
Furthermore, each infinite sum in (b) and (c) are equivalent to $\|f\|_{\dot{B}_{p,q}^{\alpha,L}}$.

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p \leq \infty$. The following three statements are equivalent for $f \in (\mathcal{M}_{p',q'}^{-\alpha,L^*})'$.

(a) $f \in \dot{B}_{p,q}^{\alpha,L}$

(b) f satisfies $\left\{ \sum_{j=-\infty}^{\infty} (2^{j\alpha} \|\Psi_{2^{-j}}(L)f\|_p)^q \right\}^{1/q} < \infty$

(c) f satisfies $\left\{ \sum_{j=-\infty}^{\infty} (2^{j\alpha} \|\Delta_j(L)f\|_p)^q \right\}^{1/q} < \infty$

where $\Delta_j(L)f = e^{-2^{-j}L}f - e^{-2^{-j-1}L}f$.
Furthermore, each infinite sum in (b) and (c) are equivalent to $\|f\|_{\dot{B}_{p,q}^{\alpha,L}}$.

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \le p \le \infty$. The following three statements are equivalent for $f \in (\mathcal{M}_{p',q'}^{-\alpha,L^*})'$.

(a) $f \in \dot{B}_{p,q}^{\alpha,L}$

(b) f satisfies $\left\{ \sum_{j=-\infty}^{\infty} (2^{j\alpha} \|\Psi_{2^{-j}}(L)f\|_p)^q \right\}^{1/q} < \infty$

(c) f satisfies $\left\{ \sum_{j=-\infty}^{\infty} (2^{j\alpha} \|\Delta_j(L)f\|_p)^q \right\}^{1/q} < \infty$

where $\Delta_j(L)f = e^{-2^{-j}L}f - e^{-2^{-j-1}L}f$.
Furthermore, each infinite sum in (b) and (c) are equivalent to $\|f\|_{\dot{B}_{p,q}^{\alpha,L}}$.

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. For any $f \in (\mathcal{M}_{p',q'}^{-\alpha,L^*})'$, and $k = 1, 2, \ldots$, we define a family of Besov norms by

$$\|f\|_{\dot{B}_{p,q}^{\alpha,L,k}} = \left\{ \int_0^\infty (t^{-\alpha} \|t^k L^k e^{-tL} f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}_{p,q}^{\alpha,L,k}} = \sup_{t>0} t^{-\alpha} \|t^k L^k e^{-tL} f\|_p$$

for $q = \infty$, where $t^k L^k e^{-tL} f(x) = (f, p_{k,t}(x, \cdot))$. Then these norms for different values of k are equivalent to each other. Also, these norms for different values of non-integer w, for $k < w < k+1$, are equivalent to each other.

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. For any $f \in \big(\mathcal{M}_{p',q'}^{-\alpha,L^*}\big)'$, and $k = 1, 2, \ldots$, we define a family of Besov norms by

$$\|f\|_{\dot{B}_{p,q}^{\alpha,L,k}} = \left\{ \int_0^\infty (t^{-\alpha}\|t^k L^k e^{-tL} f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}_{p,q}^{\alpha,L,k}} = \sup_{t>0} t^{-\alpha}\|t^k L^k e^{-tL} f\|_p$$

for $q = \infty$, where $t^k L^k e^{-tL} f(x) = (f, p_{k,t}(x, \cdot))$. Then these norms for different values of k are equivalent to each other. Also, these norms for different values of non-integer w, for $k < w < k+1$, are equivalent to each other.

Suppose L satisfies (S) and (K). Let $-1 < \alpha < 1$ and $1 \leq p, q \leq \infty$. For any $f \in \big(\mathcal{M}_{p',q'}^{-\alpha,L^*}\big)'$, and $k = 1, 2, \ldots$, we define a family of Besov norms by

$$\|f\|_{\dot{B}_{p,q}^{\alpha,L,k}} = \left\{ \int_0^\infty (t^{-\alpha}\|t^k L^k e^{-tL} f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}_{p,q}^{\alpha,L,k}} = \sup_{t>0} t^{-\alpha}\|t^k L^k e^{-tL} f\|_p$$

for $q = \infty$, where $t^k L^k e^{-tL} f(x) = (f, p_{k,t}(x, \cdot))$. Then these norms for different values of k are equivalent to each other. Also, these norms for different values of non-integer w, for $k < w < k + 1$, are equivalent to each other.

The next result gives the equivalence of Besov norms of more general class of functions $\Psi_t(L)$ with suitable decay at 0 and infinity.

Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$ and $1 \le p, q \le \infty$. For any $f \in (\mathcal{M}_{p',q'}^{-\alpha,L^*})'$, we define a family of Besov norms by

$$\|f\|_{\dot{B}_{p,q}^{\alpha,\Psi_t(L)}} = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}_{p,q}^{\alpha,\Psi_t(L)}} = \sup_{t>0} t^{-\alpha}\|\Psi_t(L)f\|_p$$

for $q = \infty$.

The next result gives the equivalence of Besov norms of more general class of functions $\Psi_t(L)$ with suitable decay at 0 and infinity.

Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$ and $1 \leq p, q \leq \infty$. For any $f \in (\mathcal{M}_{p',q'}^{-\alpha,L^*})'$, we define a family of Besov norms by

$$\|f\|_{\dot{B}_{p,q}^{\alpha,\Psi_t(L)}} = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}_{p,q}^{\alpha,\Psi_t(L)}} = \sup_{t>0} t^{-\alpha}\|\Psi_t(L)f\|_p$$

for $q = \infty$.

The next result gives the equivalence of Besov norms of more general class of functions $\Psi_t(L)$ with suitable decay at 0 and infinity.

Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$ and $1 \leq p, q \leq \infty$. For any $f \in \left(\mathcal{M}_{p',q'}^{-\alpha,L^*}\right)'$, we define a family of Besov norms by

$$\|f\|_{\dot{B}_{p,q}^{\alpha,\Psi_t(L)}} = \left\{ \int_0^\infty (t^{-\alpha}\|\Psi_t(L)f\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

for $q < \infty$ and

$$\|f\|_{\dot{B}_{p,q}^{\alpha,\Psi_t(L)}} = \sup_{t>0} t^{-\alpha}\|\Psi_t(L)f\|_p$$

for $q = \infty$.

MACQUARIE UNIVERSITY

Assume that $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ are two classes of functions of L which satisfy the following conditions:

- $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ are holomorphic functions on the positive x-axis such that $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ tend to 0 as ξ tends to 0 and as ξ tends to infinity.
- The operators $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ have kernel bounds (K).
- There exists $\tilde{\tilde{\Psi}}_t(L)$ with kernel bounds (K) such that

$$\tilde{\Psi}_t(L) = \Psi_t(L)\tilde{\tilde{\Psi}}_t(L).$$

Assume that $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ are two classes of functions of L which satisfy the following conditions:

- $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ are holomorphic functions on the positive x-axis such that $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ tend to 0 as ξ tends to 0 and as ξ tends to infinity.
- The operators $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ have kernel bounds (K).
- There exists $\tilde{\tilde{\Psi}}_t(L)$ with kernel bounds (K) such that

$$\tilde{\Psi}_t(L) = \Psi_t(L)\tilde{\tilde{\Psi}}_t(L).$$

Assume that $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ are two classes of functions of L which satisfy the following conditions:

- $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ are holomorphic functions on the positive x-axis such that $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ tend to 0 as ξ tends to 0 and as ξ tends to infinity.
- The operators $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ have kernel bounds (K).
- There exists $\tilde{\tilde{\Psi}}_t(L)$ with kernel bounds (K) such that

$$\tilde{\Psi}_t(L) = \Psi_t(L)\tilde{\tilde{\Psi}}_t(L).$$

MACQUARIE UNIVERSITY

- The functions $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ satisfy for some constant C

$$\tilde{\Psi}_t(L) = Ct\frac{d}{dt}(\Psi_t(L)).$$

Then the Besov norms with respect to $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ are equivalent to each other.

- The functions $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ satisfy for some constant C

$$\tilde{\Psi}_t(L) = Ct\frac{d}{dt}(\Psi_t(L)).$$

Then the Besov norms with respect to $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ are equivalent to each other.

- The functions $\Psi(\xi)$ and $\tilde{\Psi}(\xi)$ satisfy for some constant C

$$\tilde{\Psi}_t(L) = Ct\frac{d}{dt}(\Psi_t(L)).$$

Then the Besov norms with respect to $\Psi_t(L)$ and $\tilde{\Psi}_t(L)$ are equivalent to each other.

The following result gives norm equivalence for the Besov spaces with positive α.

Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$ and $1 \leq p, q \leq \infty$. A functional f belongs to $\dot{B}^{\alpha,L}_{p,q}$ if and only if f satisfies

$$\left\{ \int_0^\infty (t^{-\alpha} \|(I - e^{-tL})f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} < \infty.$$

Furthermore, the above expression is equivalent to $\|f\|_{\dot{B}^{\alpha,L}_{p,q}}$.

The following result gives norm equivalence for the Besov spaces with positive α.

Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$ and $1 \leq p, q \leq \infty$. A functional f belongs to $\dot{B}^{\alpha,L}_{p,q}$ if and only if f satisfies

$$\left\{ \int_0^\infty (t^{-\alpha} \|(I - e^{-tL})f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} < \infty.$$

Furthermore, the above expression is equivalent to $\|f\|_{\dot{B}^{\alpha,L}_{p,q}}$.

The following result gives norm equivalence for the Besov spaces with positive α.

Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$ and $1 \le p, q \le \infty$. A functional f belongs to $\dot{B}^{\alpha,L}_{p,q}$ if and only if f satisfies

$$\left\{ \int_0^\infty (t^{-\alpha} \|(I - e^{-tL})f\|_p)^q \, \frac{dt}{t} \right\}^{1/q} < \infty.$$

Furthermore, the above expression is equivalent to $\|f\|_{\dot{B}^{\alpha,L}_{p,q}}$.

Let $0 < \alpha < 1, 0 < \gamma < 1$ and $\alpha + \gamma < 1$. For any $f \in \dot{B}^{\alpha,L}_{p,q}$, we define fractional integrals $L^{-\gamma} f$ associated with an operator L by

$$(L^{-\gamma} f, \phi) = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} (e^{-tL} f, \phi)\, dt$$

for every $\phi \in \mathcal{M}^{-(\alpha+\gamma),L^*}_{p',q'}$, where $\Gamma(\gamma)$ is an appropriate constant.

If we let L be the Laplacian $-\Delta$ on $\mathbb{R}^n$, then $L^{-\gamma}$ will be the classical fractional integral.

Let $0 < \alpha < 1, 0 < \gamma < 1$ and $\alpha + \gamma < 1$. For any $f \in \dot{B}_{p,q}^{\alpha,L}$, we define fractional integrals $L^{-\gamma} f$ associated with an operator L by

$$(L^{-\gamma} f, \phi) = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} (e^{-tL} f, \phi)\, dt$$

for every $\phi \in \mathcal{M}_{p',q'}^{-(\alpha+\gamma),L^*}$, where $\Gamma(\gamma)$ is an appropriate constant.

If we let L be the Laplacian $-\Delta$ on $\mathbb{R}^n$, then $L^{-\gamma}$ will be the classical fractional integral.

Let $0 < \alpha < 1, 0 < \gamma < 1$ and $\alpha + \gamma < 1$. For any $f \in \dot{B}^{\alpha,L}_{p,q}$, we define fractional integrals $L^{-\gamma} f$ associated with an operator L by

$$(L^{-\gamma} f, \phi) = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} (e^{-tL} f, \phi)\, dt$$

for every $\phi \in \mathcal{M}^{-(\alpha+\gamma),L^*}_{p',q'}$, where $\Gamma(\gamma)$ is an appropriate constant.

If we let L be the Laplacian $-\Delta$ on $\mathbb{R}^n$, then $L^{-\gamma}$ will be the classical fractional integral.

We define the norm

$$\|L^{-\gamma} f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}} = \left\{ \int_0^\infty (t^{-(\alpha+\gamma)} \|tLe^{-tL}(L^{-\gamma} f)\|_p)^q \, \frac{dt}{t} \right\}^{1/q}$$

when the last integral is finite.

Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$, $0 < \gamma < 1$, $\alpha + \gamma < 1$ and $1 \le p, q \le \infty$.

Then there exists a positive constant C such that for all $f \in \dot{B}^{\alpha,L}_{p,q}$,

$$C^{-1} \|f\|_{\dot{B}^{\alpha,L}_{p,q}} \le \|L^{-\gamma} f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}} \le C \|f\|_{\dot{B}^{\alpha,L}_{p,q}}.$$

We define the norm

$$\|L^{-\gamma}f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}} = \left\{\int_0^\infty (t^{-(\alpha+\gamma)}\|tLe^{-tL}(L^{-\gamma}f)\|_p)^q \frac{dt}{t}\right\}^{1/q}$$

when the last integral is finite.

Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$, $0 < \gamma < 1$, $\alpha + \gamma < 1$ and $1 \le p, q \le \infty$.

Then there exists a positive constant C such that for all $f \in \dot{B}^{\alpha,L}_{p,q}$,

$$C^{-1}\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \le \|L^{-\gamma}f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}} \le C\|f\|_{\dot{B}^{\alpha,L}_{p,q}}.$$

We define the norm

$$\|L^{-\gamma}f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}} = \left\{ \int_0^\infty (t^{-(\alpha+\gamma)}\|tLe^{-tL}(L^{-\gamma}f)\|_p)^q \frac{dt}{t} \right\}^{1/q}$$

when the last integral is finite.

Suppose L satisfies (S) and (K). Let $0 < \alpha < 1$, $0 < \gamma < 1$, $\alpha + \gamma < 1$ and $1 \leq p, q \leq \infty$.

Then there exists a positive constant C such that for all $f \in \dot{B}^{\alpha,L}_{p,q}$,

$$C^{-1}\|f\|_{\dot{B}^{\alpha,L}_{p,q}} \leq \|L^{-\gamma}f\|_{\dot{B}^{\alpha+\gamma,L}_{p,q}} \leq C\|f\|_{\dot{B}^{\alpha,L}_{p,q}}.$$

Decomposition of Besov spaces associated with operators

Suppose that V is a fixed non-negative function on $\mathbb{R}^n$, $n \geq 3$, satisfying a *reverse Hölder inequality* $RH_S(\mathbb{R}^n)$ for some $s > \frac{n}{2}$; that is, there is a $C = C(s, V) > 0$ with the property that

$$\Big(\frac{1}{|B|}\int_B V(x)^s\,dx\Big)^{1/s} \leq \frac{C}{|B|}\int_B V(x)\,dx$$

for all balls $B \subset \mathbb{R}^n$.

We consider the time independent Schrödinger operator with the potential V on $L^2(\mathbb{R}^n)$:

$$L = -\Delta + V(x).$$

Decomposition of Besov spaces associated with operators

Suppose that V is a fixed non-negative function on $\mathbb{R}^n$, $n \geq 3$, satisfying a *reverse Hölder inequality* $RH_S(\mathbb{R}^n)$ for some $s > \frac{n}{2}$; that is, there is a $C = C(s, V) > 0$ with the property that

$$\Big(\frac{1}{|B|}\int_B V(x)^s\,dx\Big)^{1/s} \leq \frac{C}{|B|}\int_B V(x)\,dx$$

for all balls $B \subset \mathbb{R}^n$.

We consider the time independent Schrödinger operator with the potential V on $L^2(\mathbb{R}^n)$:

$$L = -\Delta + V(x).$$

Decomposition of Besov spaces associated with operators

Suppose that V is a fixed non-negative function on $\mathbb{R}^n$, $n \geq 3$, satisfying a *reverse Hölder inequality* $RH_S(\mathbb{R}^n)$ for some $s > \frac{n}{2}$; that is, there is a $C = C(s, V) > 0$ with the property that

$$\Big(\frac{1}{|B|} \int_B V(x)^s \, dx \Big)^{1/s} \leq \frac{C}{|B|} \int_B V(x) \, dx$$

for all balls $B \subset \mathbb{R}^n$.

We consider the time independent Schrödinger operator with the potential V on $L^2(\mathbb{R}^n)$:

$$L = -\Delta + V(x).$$

In the following, the definition of a molecule associated with a cube $Q = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}$ involves the "lower left corner of Q", $x_Q = a = (a_1, a_2, \ldots, a_n)$, and $\ell(Q)$, the side length of Q.

Let $\epsilon \in (0, 1]$. A function m_Q is called an (ϵ, L, Q)-molecule if $m_Q = Lg_Q$ for some g_Q, and the following conditions hold:

$$|m_Q(x)| + \ell(Q)^{-2}|g_Q(x)| \leq |Q|^{-1}\left\{1 + \frac{|x - x_Q|}{\ell(Q)}\right\}^{-n-\epsilon} \quad \text{for } x \in \mathbb{R}^n;$$

$$\int_{|y| \leq \ell(Q)} \|m_Q(x+y) - m_Q(x)\|_{L^1(dx)} \frac{dy}{|y|^n} \leq 1.$$

In the following, the definition of a molecule associated with a cube $Q = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}$ involves the "lower left corner of Q", $x_Q = a = (a_1, a_2, \ldots, a_n)$, and $\ell(Q)$, the side length of Q.

Let $\epsilon \in (0, 1]$. A function m_Q is called an (ϵ, L, Q)-molecule if $m_Q = Lg_Q$ for some g_Q, and the following conditions hold:

$$|m_Q(x)| + \ell(Q)^{-2}|g_Q(x)| \leq |Q|^{-1}\left\{1 + \frac{|x - x_Q|}{\ell(Q)}\right\}^{-n-\epsilon} \quad \text{for } x \in \mathbb{R}^n;$$

$$\int_{|y| \leq \ell(Q)} \|m_Q(x + y) - m_Q(x)\|_{L^1(dx)} \frac{dy}{|y|^n} \leq 1.$$

In the following, the definition of a molecule associated with a cube $Q = \{x \in \mathbb{R}^n : a_i \le x_i \le b_i, i = 1, 2, \dots, n\}$ involves the "lower left corner of Q", $x_Q = a = (a_1, a_2, \dots, a_n)$, and $\ell(Q)$, the side length of Q.

Let $\epsilon \in (0, 1]$. A function m_Q is called an (ϵ, L, Q)-molecule if $m_Q = Lg_Q$ for some g_Q, and the following conditions hold:

$$|m_Q(x)| + \ell(Q)^{-2}|g_Q(x)| \le |Q|^{-1}\left\{1 + \frac{|x - x_Q|}{\ell(Q)}\right\}^{-n-\epsilon} \quad \text{for } x \in \mathbb{R}^n;$$

$$\int_{|y| \le \ell(Q)} \|m_Q(x+y) - m_Q(x)\|_{L^1(dx)} \frac{dy}{|y|^n} \le 1.$$

Decomposition of Besov spaces associated with operators

Molecular decomposition of $\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$

Suppose that $L = -\Delta + V$, where $V \not\equiv 0$ is a non-negative potential in $RH_s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$. Assume that $f \in L^1(\mathbb{R}^n)$. The following are equivalent properties of f:

(i) $f \in \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$.

(ii) For any $0 < \epsilon \leq 1$, there exist a sequence of coefficients $\{s_Q\}$, $0 \leq s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence $\{m_Q\}$ of (ϵ, L, Q)-molecules such that

$$f = \sum_Q s_Q m_Q \qquad \text{in} \quad \dot{B}^{0,L}_{1,1}(\mathbb{R}^n),$$

and $\sum_Q |s_Q| \leq C\|f\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)}$.

Suppose that $L = -\Delta + V$, where $V \not\equiv 0$ is a non-negative potential in $RH_s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$. Assume that $f \in L^1(\mathbb{R}^n)$. The following are equivalent properties of f:

(i) $f \in \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$.

(ii) For any $0 < \epsilon \leq 1$, there exist a sequence of coefficients $\{s_Q\}$, $0 \leq s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence $\{m_Q\}$ of (ϵ, L, Q)-molecules such that

$$f = \sum_Q s_Q m_Q \qquad \text{in} \quad \dot{B}^{0,L}_{1,1}(\mathbb{R}^n),$$

and $\sum_Q |s_Q| \leq C\|f\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)}$.

Suppose that $L = -\Delta + V$, where $V \not\equiv 0$ is a non-negative potential in $RH_s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$. Assume that $f \in L^1(\mathbb{R}^n)$. The following are equivalent properties of f:

(i) $f \in \dot{B}^{0,L}_{1,1}(\mathbb{R}^n)$.

(ii) For any $0 < \epsilon \leq 1$, there exist a sequence of coefficients $\{s_Q\}$, $0 \leq s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence $\{m_Q\}$ of (ϵ, L, Q)-molecules such that

$$f = \sum_Q s_Q m_Q \qquad \text{in} \quad \dot{B}^{0,L}_{1,1}(\mathbb{R}^n),$$

and $\sum_Q |s_Q| \leq C\|f\|_{\dot{B}^{0,L}_{1,1}(\mathbb{R}^n)}$.

A. Wong, Molecular Decomposition of Besov Spaces Associated With Schrödinger Operators, *Commun. Math. Anal.*, **16** (2014), 48–56.

The following result is a molecular characterization of $\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$.

In the following, the definition of a molecule associated with a cube $Q = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}$ involves the "lower left corner of Q", $x_Q = a = (a_1, a_2, \ldots, a_n)$, and $\ell(Q)$, the side length of Q. Given $j \in \mathbb{Z}$, we use $\mathbb{D}_j$ to denote the set of all dyadic cubes of sidelength 2^{-j}.

A. Wong, Molecular Decomposition of Besov Spaces Associated With Schrödinger Operators, *Commun. Math. Anal.*, **16** (2014), 48–56.

The following result is a molecular characterization of $\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$.

In the following, the definition of a molecule associated with a cube $Q = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}$ involves the "lower left corner of Q", $x_Q = a = (a_1, a_2, \ldots, a_n)$, and $\ell(Q)$, the side length of Q. Given $j \in \mathbb{Z}$, we use $\mathbb{D}_j$ to denote the set of all dyadic cubes of sidelength 2^{-j}.

MACQUARIE UNIVERSITY

A. Wong, Molecular Decomposition of Besov Spaces Associated With Schrödinger Operators, *Commun. Math. Anal.*, **16** (2014), 48–56.

The following result is a molecular characterization of $\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$.

In the following, the definition of a molecule associated with a cube $Q = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}$ involves the "lower left corner of Q", $x_Q = a = (a_1, a_2, \ldots, a_n)$, and $\ell(Q)$, the side length of Q. Given $j \in \mathbb{Z}$, we use $\mathbb{D}_j$ to denote the set of all dyadic cubes of sidelength 2^{-j}.

MACQUARIE UNIVERSITY

Let $\epsilon \in (0,1]$, $\alpha \in (-1,1)$ and $p \geq 1$. A function m_Q is called an (ϵ, α, p)-molecule for L associated to the cube Q if $m_Q = Lg_Q$ for some g_Q, and the following conditions hold:

$$|m_Q(x)| + \ell(Q)^{-2}|g_Q(x)| \leq \ell(Q)^{\alpha - n/p}\left\{1 + \frac{|x - x_Q|}{\ell(Q)}\right\}^{-n-\epsilon} \quad \text{for } x \in \mathbb{R}^n;$$

$$\int_{|y| \leq \ell(Q)} \|m_Q(x+y) - m_Q(x)\|_{L^p(dx)} \frac{dy}{|y|^{n+\alpha}} \leq 1.$$

Let $\epsilon \in (0,1]$, $\alpha \in (-1,1)$ and $p \geq 1$. A function m_Q is called an (ϵ, α, p)-molecule for L associated to the cube Q if $m_Q = Lg_Q$ for some g_Q, and the following conditions hold:

$$|m_Q(x)| + \ell(Q)^{-2}|g_Q(x)| \leq \ell(Q)^{\alpha - n/p}\left\{1 + \frac{|x - x_Q|}{\ell(Q)}\right\}^{-n-\epsilon} \quad \text{for } x \in \mathbb{R}^n;$$

$$\int_{|y| \leq \ell(Q)} \|m_Q(x+y) - m_Q(x)\|_{L^p(dx)} \frac{dy}{|y|^{n+\alpha}} \leq 1.$$

Suppose that $L = -\Delta + V$, where $V \not\equiv 0$ is a non-negative potential in $RH_s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$. Assume that $f \in L^1(\mathbb{R}^n)$ and let σ be a certain constant. Let $-1 < \alpha < \min\{1, \sigma\}$ and $1 \leq p \leq q < \infty$. Then the following are equivalent:

(a) $f \in \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$.

(b) For any $0 < \epsilon \leq 1$, there exist a sequence of coefficients $\{s_Q\}$, $0 \leq s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence $\{m_Q\}$ of (ϵ, α, p)-molecules for L, such that

$$f = \sum_Q s_Q m_Q \qquad \text{in} \quad \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n),$$

and $\left(\sum_{j\in\mathbb{Z}}\left(\sum_{Q\in\mathbb{D}_j} |s_Q|^p\right)^{q/p}\right)^{1/q} \approx \|f\|_{\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)}$.

Suppose that $L = -\Delta + V$, where $V \not\equiv 0$ is a non-negative potential in $RH_s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$. Assume that $f \in L^1(\mathbb{R}^n)$ and let σ be a certain constant. Let $-1 < \alpha < \min\{1, \sigma\}$ and $1 \leq p \leq q < \infty$. Then the following are equivalent:

(a) $f \in \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$.

(b) For any $0 < \epsilon \leq 1$, there exist a sequence of coefficients $\{s_Q\}$, $0 \leq s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence $\{m_Q\}$ of (ϵ, α, p)-molecules for L, such that

$$f = \sum_Q s_Q m_Q \qquad \text{in} \quad \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n),$$

and $\left(\sum_{j\in\mathbb{Z}}\left(\sum_{Q\in\mathbb{D}_j} |s_Q|^p\right)^{q/p}\right)^{1/q} \approx \|f\|_{\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)}$.

MACQUARIE UNIVERSITY

Suppose that $L = -\Delta + V$, where $V \not\equiv 0$ is a non-negative potential in $RH_s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$. Assume that $f \in L^1(\mathbb{R}^n)$ and let σ be a certain constant. Let $-1 < \alpha < \min\{1, \sigma\}$ and $1 \leq p \leq q < \infty$. Then the following are equivalent:

(a) $f \in \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$.

(b) For any $0 < \epsilon \leq 1$, there exist a sequence of coefficients $\{s_Q\}$, $0 \leq s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence $\{m_Q\}$ of (ϵ, α, p)-molecules for L, such that

$$f = \sum_Q s_Q m_Q \qquad \text{in} \quad \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n),$$

and $\left(\sum_{j\in\mathbb{Z}}\left(\sum_{Q\in\mathbb{D}_j} |s_Q|^p\right)^{q/p}\right)^{1/q} \approx \|f\|_{\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)}$.

Summary

- We studied the theory of Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Gaussian kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth.
- We studied an embedding theorem for the Besov spaces and gave discrete characterizations of the Besov norms associated to operators.

MACQUARIE UNIVERSITY

Summary

- We studied the theory of Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Gaussian kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth.
- We studied an embedding theorem for the Besov spaces and gave discrete characterizations of the Besov norms associated to operators.

MACQUARIE UNIVERSITY

Summary

- We studied the theory of Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Gaussian kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth.
- We studied an embedding theorem for the Besov spaces and gave discrete characterizations of the Besov norms associated to operators.

Summary

- Depending on the choice of L, the Besov spaces are natural settings for generic estimates for certain singular integral operators such as the fractional powers L^α.
- As an application, we studied the decomposition of Besov spaces associated with Schrödinger operators with non-negative potentials satisfying reverse Hölder estimates.

MACQUARIE UNIVERSITY

Summary

- Depending on the choice of L, the Besov spaces are natural settings for generic estimates for certain singular integral operators such as the fractional powers L^α.
- As an application, we studied the decomposition of Besov spaces associated with Schrödinger operators with non-negative potentials satisfying reverse Hölder estimates.

MACQUARIE UNIVERSITY

Summary

- Depending on the choice of L, the Besov spaces are natural settings for generic estimates for certain singular integral operators such as the fractional powers L^α.
- As an application, we studied the decomposition of Besov spaces associated with Schrödinger operators with non-negative potentials satisfying reverse Hölder estimates.

MACQUARIE UNIVERSITY

Thank you