Harmonic Analysis of Grushin Type Operators

Adam Sikora

Macquarie University

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Abstract

We consider the classical Grushin operator $\partial_x^2 + x^2\partial_y^2$ and its natural generalization. For this class of degenerate elliptic operators we study some standard harmonic analysis type problems like heat kernel bounds, Poincar inequalities, Riesz transform, convergence of eignefunction expansions, spectral multipliers or Bochner-Riesz type analysis.

The talk is based on a range of results obtained by Chen, Martini, Müller, Ouhabaz, Robinson and myself.

Class of Grushin type operators

We consider degenerate, second-order, elliptic operators H in divergence form on $L_2(\mathbb{R}^n \times \mathbb{R}^m)$. We assume the coefficients are real symmetric and $a_1 H_\delta \geq H \geq a_2 H_\delta$ for some $a_1, a_2 > 0$ where

$$H_\delta = -\nabla_{x_1} \, a_{\delta_1,\delta_1'}(x_1) \, \nabla_{x_1} - a_{\delta_2,\delta_2'}(x_1) \, \nabla_{x_2}^2 \quad .$$

Here $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$ and $a_{\delta_i,\delta_i'}$ are positive measurable functions such that $a_{\delta_i,\delta_i'}(x)$ behaves like $|x|^{\delta_i}$ as $x \to 0$ and $|x|^{\delta_i'}$ as $x \to \infty$ with $\delta_1, \delta_1' \in [0, 2\rangle$ and $\delta_2, \delta_2' \geq 0$.

Classical Grushin operators

Original definition of Grushin (Grušin?) operators

$$H = -(\partial_x^2 + x^{2k}\partial_y^2)$$

Classical Grushin operators

Main point of interest

- degenerate elliptic operator (not uniformly elliptic) nevertheless H is fully subelliptic.
- Homogenous space (with the doubling condition)

$$B(x,r) \leq CB(x,2r)$$

  But not "too homogenous" the volume $V(B(x,r))$ depends essentially not only on r but also on a space variable x.
- Interesting structure "skew Cartesian product" on $(x,y) \in M_1 \times M_2$

$$\Delta = \Delta_x + g(x)^2 \Delta_y$$

  similar to conic manifolds or connected copies of $\mathbb{R}^n$.

One may associate with the coefficients $A = (a_{ij})$ a 'distance' $d_A \colon \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ by setting

$$d_A(x\,;y) = \sup_{\psi \in \mathcal{D}} |\psi(x) - \psi(y)| \tag{1}$$

for all $x, y \in \mathbb{R}^d$, where

$$\mathcal{D} = \{\psi \in C_c^\infty(\mathbb{R}^d) : \psi \text{ real and } \Big\| \sum_{i,j=1}^{d} a_{ij}\,(\partial_i\psi)\,(\partial_j\psi)\Big\|_\infty \le 1\}\,.$$

The doubling condition - yes

Gaussian bounds

- Robinson, Sikora, Math. Z. '08

Theorem

For each $\varepsilon > 0$ there is an $a > 0$ such that the semigroup kernel K of the Grushin operator satisfies

$$0 \le K_t(x\,;y) \le a\,(|B(x\,;t^{1/2})|\,|B(y\,;t^{1/2})|)^{-1/2}\, e^{-d(x;y)^2/(4(1+\varepsilon)t)} \tag{2}$$

for all $t > 0$ and almost all $x, y \in \mathbb{R}^{n+m}$.

As a Corollary one gets Alexopoulos type spectral multipliers and boudedness of Riesz transform for $1 < p \le 2$ (by Coulhon-Duong). **However the critical exponent for spectral multipliers is an open problem and Riesz transform for $p > 2$.**

Poincaré inequality

$\Gamma(\psi) = \sum_{i,j=1}^{d} a_{ij}(x,y)(\partial_i\psi)(\partial_j\psi)$ denotes the *carré du champ* (gradient) associated with H.

- Robinson, Sikora, Math. '13

Theorem

I. *If $n \geq 2$, or if $n = 1$ and $\delta_1 \vee \delta_1' \in [0, 1/2\rangle$, then then there exist $\lambda > 0$ and $\kappa \in \langle 0, 1]$ such that*

$$\int_{B(x;r)} dy\, \Gamma(\varphi)(y) \geq \lambda\, r^{-2} \int_{B(x;\kappa r)} dy\, (\varphi(y) - \langle\varphi\rangle_B)^2 \qquad (3)$$

for all $x \in \mathbb{R}^{n+m}$, $r > 0$ and $\varphi \in C^1(\mathbb{R}^{n+m})$ where $\langle\varphi\rangle_B = |B(x\,;\kappa r)|^{-1} \int_{B(x;\kappa r)} dy\, \varphi(y)$.

Poincaré inequality

Theorem

II. *If* $n = 1$ *and* $\delta_1 \vee \delta_1' \in [1/2, 1\rangle$ *then the uniform Poincaré inequality* (3) *fails.*

III. *If* $n = 1$ *and* $\delta_1 \in [1/2, 1\rangle$ *then there then there exist* $\lambda > 0$ *and* $\kappa \in \langle 0, 1]$ *such that*

$$\int_{B_\pm(x;r)} dy\, \Gamma(\varphi)(y) \geq \lambda\, r^{-2} \int_{B_\pm(x;\kappa r)} dy\, \big(\varphi(y) - \langle \varphi \rangle_{B_\pm}\big)^2 \qquad (4)$$

for all $x \in \Omega_\pm$, $r > 0$ *and* $\varphi \in C^1(\mathbb{R}^{1+m})$.

$\Omega_+ = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^m : x_1 \geq 0\}$,
$\Omega_- = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^m : x_1 \leq 0\}$ and then define 'balls' $B_\pm(x\,;r)$ by $B_\pm(x\,;r) = B(x\,;r) \cap \Omega_\pm$.

Sharp spectral multipliers - roughly speaking the same problem as sharp Bochner-Riesz summability
Motivated by the analogue of the result for the sublaplacian on the Heisenberg group obtained by D. Müller and E.M. Stein and by W. Hebisch. We consider the class

$$H = -\Delta_x - |x|^2 \Delta_y,$$

acting on $\mathbb{R}^m \times \mathbb{R}^n$

Sharp spectral multipliers

- Martini, Sikora '12

Set $D = \max\{n+m, 2n\}$

Theorem

Suppose that $\kappa > (D-1)/2$ and $p \in [1,\infty]$. Then the Bochner-Riesz means $(1-tH)_+^\kappa$ are bounded on $L^p(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ uniformly in $0 < t < \infty$.

- Martini, Müller '13 $D = n+m$ is OK. (And more results of this type for Nilpotent Lie groups etc.)

$$L_\sigma = -\sum_{j=1}^{m} \partial_{x_j}^2 - \left(\sum_{j=1}^{m} |x_j|^\sigma\right) \sum_{k=1}^{n} \partial_{y_k}^2 \tag{5}$$

$\sigma = 1$ (before $\sigma = 2$)

- Chen, Sikora '13

Set $D = \max\{m+n, 3n/2\}$

Theorem

Suppose that $\kappa > (D-1)/2$ and $p \in [1, \infty]$. Then the Bochner-Riesz means $(1 - tH)_+^\kappa$ are bounded on $L^p(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ uniformly in $0 < t < \infty$.

Sharp spectral multipliers for $\sigma = 2$ but L^p for $p > 1$

- Chen, Ouhabaz '14

Back to Set $D = \max\{n+m, 2n\}$

Theorem

Let $1 \leq p \leq \min\{2m/(m+2), (2n+2)/(n+3)\}$. Suppose that $\delta > \max\{D|1/p - 1/2| - 1/2, 0\}$. Then the Bochner-Riesz means $(1 - tL)_+^\delta$ are bounded on $L^p(\mathbf{R}^{d_1} \times \mathbf{R}^{d_2})$ uniformly in $t \in [0, \infty)$.

The proof based on H. Koch and D. Tataru, L^p eigenfunction bounds for the Hermite operator.

Riesz transform L^p for $p > 2$

- Robinson, Sikora '14

For $\sigma = 2N$ Riesz transform is bounded for on L^p for all $1 < p < \infty$.
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