Thin-film limit of the Navier–Stokes equations in a curved thin domain

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Navier–Stokes (NS) eqs. in a curved thin domain

- Γ: given 2D closed surface in $\mathbb{R}^3$ (e.g. sphere, torus)
- $\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid \mathrm{dist}(x,\Gamma) < \varepsilon/2\}$ ($\varepsilon > 0$: small)
- NS eqs. with Navier's (perfect) slip B.C. in Ω_ε

$$(NS_\varepsilon)\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla p^\varepsilon = \nu\Delta u & \text{in} \quad \Omega_\varepsilon \times (0,\infty) \\ \mathrm{div}\, u^\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon \times (0,\infty) \\ u^\varepsilon \cdot n_\varepsilon = 0,\ 2\nu P_\varepsilon D(u^\varepsilon)n_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon \times (0,\infty) \\ u^\varepsilon|_{t=0} = u_0^\varepsilon & \text{in} \quad \Omega_\varepsilon \end{cases}$$

  - $\nu > 0$: viscosity coefficient independent of ε
  - n_ε: unit outer normal of $\partial\Omega_\varepsilon$
  - $P_\varepsilon = I_3 - n_\varepsilon \otimes n_\varepsilon$, $2D(u^\varepsilon) = \nabla u^\varepsilon + (\nabla u^\varepsilon)^T$
- Aim: study the behavior of u^ε as $\varepsilon \to 0$ and derive limit eqs.

Previous works on NS eqs. in thin domains

Main problems in the study of the NS eqs. in 3D thin domains

- Global existence of a strong solution $\boldsymbol{u}^\varepsilon$ for large data
- Convergence of $\boldsymbol{u}^\varepsilon$ as $\varepsilon \to 0$ in an appropriate sense
- Characterization of the limit of $\boldsymbol{u}^\varepsilon$ as a sol. to limit eqs.

Previous works

- Raugel–Sell (1993), Temam–Ziane (1996), etc.:
  $\Omega_\varepsilon = \omega \times (0,\varepsilon)$, ω: 2D domain
- Iftimie–Raugel–Sell (2007), Hoang (2010), Hoang–Sell (2010):
  $\Omega_\varepsilon = \{(x',x_3) \mid x' \in (0,1)^2,\ \varepsilon g_0(x') < x_3 < \varepsilon g_1(x')\}$
- Temam–Ziane (1997): $\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid 1 < |x| < 1+\varepsilon\}$

Our case

- Ω_ε around a general surface Γ with nonconstant curvatures

Notations

- n: unit outer normal of Γ

$$\Omega_\varepsilon = \{y + rn(y) \mid y \in \Gamma,\ r \in (-\varepsilon/2, \varepsilon/2)\}$$

- Average of $u\colon \Omega_\varepsilon \to \mathbb{R}^3$ and its tangential component

$$Mu(y) = \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} u(y + rn(y))\,dr, \quad y \in \Gamma$$

$$M_\tau u(y) = Mu(y) - \{Mu(y)\cdot n(y)\}n(y)$$

- Initial data of (NS_ε) satisfies

$$u_0^\varepsilon \in H^1(\Omega_\varepsilon)^3,\ \ \mathrm{div}\, u_0^\varepsilon = 0 \ \text{ in } \ \Omega_\varepsilon,\ \ u_0^\varepsilon \cdot n_\varepsilon = 0 \ \text{ on } \ \partial\Omega_\varepsilon$$

Main theorem

Theorem 1 (M., 2020, Adv. Diff. Equ.)

Under suitable assumptions on Γ and u_0^ε, suppose that

(a) $\exists c > 0$, $\exists \varepsilon_1, \alpha \in (0,1)$, s.t.

$$\|u_0^\varepsilon\|_{H^1(\Omega_\varepsilon)}^2 \leq c\varepsilon^{-1+\alpha}, \quad \forall \varepsilon \in (0, \varepsilon_1)$$

(b) $\exists$tangential $v_0 \in L^2(\Gamma)^3$ s.t. $\lim_{\varepsilon \to 0} M_\tau u_0^\varepsilon = v_0$ weakly in $L^2(\Gamma)^3$

Then $\exists \varepsilon_2 \in (0, \varepsilon_1)$ s.t. $\forall \varepsilon \in (0, \varepsilon_2)$, $\exists$global strong solution $u^\varepsilon \in C([0,\infty); H^1(\Omega_\varepsilon)^3) \cap L^2_{loc}([0,\infty); H^2(\Omega_\varepsilon)^3)$ to (NS_ε) and $\lim_{\varepsilon \to 0} Mu^\varepsilon \cdot n = 0$ strongly in $C([0,\infty); L^2(\Gamma))$.

$$(NS_\varepsilon) \begin{cases} \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla p^\varepsilon = \nu \Delta u^\varepsilon, \ \operatorname{div} u^\varepsilon = 0 \ \text{ in } \ \Omega_\varepsilon \\ u^\varepsilon \cdot n_\varepsilon = 0, \ 2\nu P_\varepsilon D(u^\varepsilon) n_\varepsilon = 0 \ \text{ on } \ \partial\Omega_\varepsilon \\ u^\varepsilon|_{t=0} = u_0^\varepsilon \ \text{ in } \ \Omega_\varepsilon \end{cases}$$

Theorem 1 (continued)

Moreover, ∃tangential vector field
$v \in C([0,\infty); L^2(\Gamma)^3) \cap L^2_{loc}([0,\infty); H^1(\Gamma)^3)$ s.t.

- $\forall T > 0$, $\lim_{\varepsilon \to 0} M_\tau u^\varepsilon = v$ weakly in $L^2(0,T; H^1(\Gamma)^3)$
- v is a unique weak solution to

$$(NS_0) \begin{cases} \partial_t v + \overline{\nabla}_v v + \nabla_\Gamma q = 2\nu P \mathrm{div}_\Gamma[D_\Gamma(v)] & \text{on } \Gamma \times (0,\infty) \\ \mathrm{div}_\Gamma v = 0 & \text{on } \Gamma \times (0,\infty) \\ v|_{t=0} = v_0 & \text{on } \Gamma \end{cases}$$

- ∇_Γ, div_Γ: tangential gradient and surface divergence on Γ
- $\overline{\nabla}_v v$: covariant derivative of v along itself
- $D_\Gamma(v)$: surface strain rate tensor

$$\nabla_\Gamma q = P\nabla q, \ \ \mathrm{div}_\Gamma v = \mathrm{tr}[\nabla_\Gamma v] \ \ (P = I_3 - n \otimes n)$$
$$\overline{\nabla}_v v = P(v \cdot \nabla_\Gamma)v, \ \ 2D_\Gamma(v) = P\{\nabla_\Gamma v + (\nabla_\Gamma v)^T\}P$$

Outline of our works

It took three papers to derive (NS_0):

- Part 1: J. Math. Sci. Univ. Tokyo, 29 (2022), 149–256. (108pp.)
  Basic inequalities in Ω_ε with explicit dependence on ε
- Part 2: J. Math. Fluid Mech., 23 (2021), 60pp.
  Global existence of u^ε with explicit estimates in terms of ε
- Part 3: Adv. Diff. Equ., 25 (2020), 457–626. (170pp.)
  Weak convergence of $M_\tau u^\varepsilon$ as $\varepsilon \to 0$
  and characterization of the limit as a sol. to (NS_0)

Why so long?

- We need to re-examine everything in view of dependence on ε (e.g. Sobolev and Korn ineqs., estimates of Stokes op.).
- Calculations in Ω_ε are more complicated due to curvatures of Γ, since we differentiate $u^\varepsilon(x) = u^\varepsilon(y + rn(y))$ w.r.t. $y \in \Gamma$.

Outline of the proof of Theorem 1

Step 0 Global existence and explicit estimates of a strong sol. u^ε (done in Part 2 by using results in Part 1)

Step 1 Derivation of a weak form (w.f) of $M_\tau u^\varepsilon$

$$\text{w.f. of } u^\varepsilon \text{ in } \Omega_\varepsilon \xrightarrow{\text{average in the thin direction}} \text{w.f. of } M_\tau u^\varepsilon \text{ on } \Gamma$$

Step 2 Energy estimate for $M_\tau u^\varepsilon$ with a bound indep. of ε

$$\max_{t\in[0,T]} \|M_\tau u^\varepsilon(t)\|^2_{L^2(\Gamma)} + \int_0^T \|M_\tau u^\varepsilon(t)\|^2_{H^1(\Gamma)}\,dt \le c_T$$

Step 3 Weak convergence of a subsequence & Characterization

$$M_\tau u^{\varepsilon_n} \xrightarrow{\varepsilon_n\to 0} v\text{: weak sol. to } (NS_0)$$

Step 4 Uniqueness of a weak sol. to $(NS_0) \Rightarrow M_\tau u^\varepsilon \xrightarrow{\varepsilon\to 0} v$

Step 1: Main idea for derivation of w.f. of $M_\tau u^\varepsilon$

- For $\eta \in H^1(\Gamma)^3$ with $\eta \cdot n = 0$, $\mathrm{div}_\Gamma \eta = 0$ on Γ, we take

$$\bar{\eta}(x) = \eta(y), \quad x = y + rn(y) \in \Omega_\varepsilon \ \ (y \in \Gamma)$$

as a test function for w.f. of u^ε and show

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} D(u^\varepsilon) : D(\bar{\eta})\,dx \approx \int_\Gamma D_\Gamma(M_\tau u^\varepsilon) : D_\Gamma(\eta)\,d\mathcal{H}^2$$

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} u^\varepsilon \otimes u^\varepsilon : \nabla\bar{\eta}\,dx \approx \int_\Gamma (M_\tau u^\varepsilon) \otimes (M_\tau u^\varepsilon) : \nabla_\Gamma \eta\,d\mathcal{H}^2$$

- Main idea: for a function φ on Ω_ε,

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \varphi(x)\,dx = \int_\Gamma \left(\frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} \varphi(y + rn(y))J(y,r)\,dr\right) d\mathcal{H}^2(y)$$

$$\approx \int_\Gamma M\varphi(y)\,d\mathcal{H}^2(y) \quad (J(y,r) \approx 1\text{: Jacobian})$$

- The use of local coordinates of Γ results in terrible calculations, since we deal with vector fields and their derivatives.
- Instead, we use the following formulas to carry out calculations in a fixed coordinate system of $\mathbb{R}^3$ (although still involved):
  - For $u^\varepsilon \colon \Omega_\varepsilon \to \mathbb{R}^3$ and $y \in \Gamma$,

$$\begin{aligned}\nabla_\Gamma M u^\varepsilon(y) \\ &= \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} \{I_3 - rW(y)\} P(y) \nabla u^\varepsilon(y + rn(y))\, dr\end{aligned}$$

  - For $\eta \colon \Gamma \to \mathbb{R}^3$ and $x = y + rn(y) \in \Omega_\varepsilon$,

$$\nabla \bar{\eta}(x) = \{I_3 - rW(y)\}^{-1} \nabla_\Gamma \eta(y)$$

  - $W = -\nabla_\Gamma n$: shape op. of Γ, $P = I_3 - n \otimes n$

Step 1: Why we need a strong sol. u^ε

- Resulting weak form (w.f.) of $M_\tau u^\varepsilon$ is

w.f. of $M_\tau u^\varepsilon =$ w.f. of $(NS_0) + R_\varepsilon$ (residual term)

- To estimate R_ε, we need the estimates for the strong sol. u^ε:

$$(\sharp)\begin{cases} \|u^\varepsilon(t)\|^2_{H^1(\Omega_\varepsilon)} \leq c\varepsilon^{-1+\alpha} \\ \int_0^t \|u^\varepsilon(s)\|^2_{H^2(\Omega_\varepsilon)}\,ds \leq c\varepsilon^{-1+\alpha}(1+t) \end{cases}$$

Here $\alpha \in (0,1)$ comes from $\|u_0^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq c\varepsilon^{-1+\alpha}$.

- Using $(\sharp)$, we can show $|R_\varepsilon| \leq c\varepsilon^{\alpha/4} \to 0$ ($\varepsilon \to 0$).

Estimate for the difference of solutions

Theorem 2 (M., 2020, Adv. Diff. Equ.)

Under the same assumptions as in Theorem 1, we have

$$\max_{t\in[0,T]} \|M_\tau u^\varepsilon(t) - v(t)\|_{L^2(\Gamma)}^2 + \int_0^T \|\nabla_\Gamma M_\tau u^\varepsilon(t) - \nabla_\Gamma v(t)\|_{L^2(\Gamma)}^2\,dt \leq c_T\left\{\varepsilon^{\alpha/2} + \|M_\tau u_0^\varepsilon - v_0\|_{L^2(\Gamma)}^2\right\}$$

for all $T > 0$, where $c_T > 0$ depends only on T.

Idea of proof

- take the difference of the weak forms of $M_\tau u^\varepsilon$ and v
- test $M_\tau u^\varepsilon - v$ and apply Gronwall's inequality

Limit eqs. are the surface NS eqs.

- Our limit eqs.:

$$(NS_0)\begin{cases}\partial_t v + \overline{\nabla}_v v + \nabla_\Gamma q = 2\nu P \mathrm{div}_\Gamma [D_\Gamma(v)] \text{ on } \Gamma \\ \mathrm{div}_\Gamma v = 0 \text{ on } \Gamma\end{cases}$$

  - ∇_Γ: tangential gradient, div_Γ: surface divergence
  - $\overline{\nabla}_v v$: covariant derivative, $P = I_3 - n \otimes n$
  - $D_\Gamma(v)$: surface strain rate tensor

- (NS_0) are the surface NS eqs.: we can rewrite (NS_0) as

$$\partial_t v + \overline{\nabla}_v v = P \mathrm{div}_\Gamma S_\Gamma, \quad \mathrm{div}_\Gamma v = 0 \text{ on } \Gamma$$

  - S_Γ : Boussinesq–Scriven surface stress tensor

$$S_\Gamma = -qP + (\lambda - \nu)(\mathrm{div}_\Gamma v)P + 2\nu D_\Gamma(v)$$

$(\lambda, \nu:$ surface dilatational and shear viscosity$)$

- We also note that our limit eqs.

$$(NS_0)\begin{cases}\partial_t v + \overline{\nabla}_v v = P\,\mathrm{div}_\Gamma S_\Gamma, \quad \mathrm{div}_\Gamma v = 0 \ \text{ on } \ \Gamma \\ S_\Gamma = -qP + (\lambda - \nu)(\mathrm{div}_\Gamma v)P + 2\nu D_\Gamma(v)\end{cases}$$

  appear as a part of or a special case of

  - Interface eqs. of two-phase flows
    cf. Slattery–Sagis–Oh (2007, book),
    Bothe–Prüss (2010), etc.
  - NS eqs. on an evolving surface
    cf. Koba–Liu–Giga (2017),
    Jankuhn–Olshanskii–Reusken (2018), etc.

Limit eqs. are intrinsic / NS eqs. on a manifold

- Our limit eqs.

$$(NS_0)\begin{cases}\partial_t v + \overline{\nabla}_v v + \nabla_\Gamma q = 2\nu P \mathrm{div}_\Gamma[D_\Gamma(v)] \text{ on } \Gamma \\ \mathrm{div}_\Gamma v = 0 \text{ on } \Gamma\end{cases}$$

  are described in terms of a fixed coordinate of $\mathbb{R}^3$ and matrices.

- However, when $v \cdot n = 0$ on Γ, we have

$$2P\mathrm{div}_\Gamma[D_\Gamma(v)] = \Delta_H v + \nabla_\Gamma(\mathrm{div}_\Gamma v) + 2Kv \text{ on } \Gamma$$

  - Δ_H: Hodge Laplacian, K: Gaussian curvature

- Hence (NS_0) can be written as

$$\partial_t v + \overline{\nabla}_v v + \nabla_\Gamma q = \nu(\Delta_H v + 2Kv), \ \mathrm{div}_\Gamma v = 0 \text{ on } \Gamma,$$

  which are intrinsic (i.e. depending only on 1st fundamental form).

- In fact, our limit eqs.

$$(NS_0)\begin{cases}\partial_t v + \overline{\nabla}_v v + \nabla_\Gamma q = \nu(\Delta_H v + 2Kv) \text{ on } \Gamma\\ \mathrm{div}_\Gamma v = 0 \text{ on } \Gamma\end{cases}$$

  agree with the NS eqs. on a Riemannian manifold introduced by

  - Ebin–Marsden (1970), Taylor (1992)

  and studied by many researchers:

  - Priebe (1994), Nagasawa (1999), Mitrea–Taylor (2001), Khesin–Misiołek (2012), Chan–Czubak (2013), Pierfelice (2017), Prüss–Simonett–Wilke (2020), etc.

- In a higher dimensional case, the Gaussian curvature K in (NS_0) is replaced by the Ricci curvature.

Limit eqs. derived under different B.C.

- Temam–Ziane (1997) studied the NS eqs. in

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3 \mid 1 < |x| < 1+\varepsilon\} \xrightarrow{\varepsilon\to 0} S^2 : \text{unit sphere}$$

Hodge B.C.: $u^\varepsilon \cdot n_\varepsilon = 0, \ \mathrm{curl}\, u^\varepsilon \times n_\varepsilon = 0$ on $\partial\Omega_\varepsilon$

to derive limit eqs. on S^2 of the form

$$\partial_t v + \overline{\nabla}_v v + \nabla_\Gamma q = \nu \Delta_H v, \ \mathrm{div}_\Gamma v = 0 \ \text{ on } \ S^2$$

- In our work, under

Slip B.C.: $u^\varepsilon \cdot n_\varepsilon = 0, \ 2\nu P_\varepsilon D(u^\varepsilon) n_\varepsilon = 0$ on $\partial\Omega_\varepsilon$

our limit eqs. (NS_0) on S^2 (with $K \equiv 1$) are of the form

$$\partial_t v + \overline{\nabla}_v v + \nabla_\Gamma q = \nu(\Delta_H v + 2v), \ \mathrm{div}_\Gamma v = 0 \ \text{ on } \ S^2$$

	B.C. on $\partial\Omega_\varepsilon$	Visc. on S^2
M.	$2\nu P_\varepsilon D(u^\varepsilon)n_\varepsilon = 0$	$\Delta_H v + 2v$
Temam–Ziane	$\mathrm{curl}\, u^\varepsilon \times n_\varepsilon = 0$	$\Delta_H v$

Difference $2v$ comes from B.C. of (NS_ε) and the curvatures of $\partial\Omega_\varepsilon$:

- Under the condition $u^\varepsilon \cdot n_\varepsilon = 0$ on $\partial\Omega_\varepsilon$, we have

$$2P_\varepsilon D(u^\varepsilon)n_\varepsilon - \mathrm{curl}\, u^\varepsilon \times n_\varepsilon = 2W_\varepsilon u^\varepsilon \quad \text{on} \quad \partial\Omega_\varepsilon$$

  - W_ε : shape operator of $\partial\Omega_\varepsilon$ (representing curvatures)

- When $\partial\Omega_\varepsilon = \{|x| = 1, 1+\varepsilon\}$, we have $W_\varepsilon u^\varepsilon \approx \pm u^\varepsilon$ and

$$2P_\varepsilon D(u^\varepsilon)n_\varepsilon - \mathrm{curl}\, u^\varepsilon \times n_\varepsilon \approx \pm 2u^\varepsilon \quad \text{on} \quad \partial\Omega_\varepsilon$$

- This $2u^\varepsilon$ results in the difference $2v$ in the two limit eqs.

Thank you for your attention!