Maximal regularity in Besov–Morrey spaces and its application to two-dimensional Keller–Segel system

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joint work with Yoshihiro Sawano

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Abstract

- In this talk, we discuss the maximal regularity of the heat equation in Besov–Morrey spaces
- Our main tools are the Fourier multipliers, which are used instead of interpolation.
- As an application, the Cauchy problems for Keller–Segel system are studied.
- This is a joint work with Yoshihiro Sawano.

Contents

Contents

We cosider the following semi-linear elliptic parabolic system.
Let $u(t,x)$ and $\psi(t,x)\ :\ \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, \quad t > 0, x \in \mathbb{R}^2, \\ -\Delta \psi = \kappa u, \quad t > 0, x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^2 \end{cases} \tag{1.1}$$

with $\kappa = \pm 1$.
$\kappa = 1 \Rightarrow$ Keller–Segel equation.
$\kappa = -1 \Rightarrow$ mono-polar drift-diffusion system for the semi-conductor simulation.

Remark 1.1

The equation (1.1) can be reduced to the corresponding integral equations:

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathrm{div}(u(s)\nabla\psi(s))\mathrm{d}s \quad (t > 0). \tag{1.2}$$

Introduction

Theorem 1.2 (N., Sawano)

Let $\kappa = \pm 1$. *Let* $1 \leq q \leq p < 2$. *Define* $\delta \equiv 2 - \dfrac{2}{p}$. *Write* $I = [0, T)$.
Then for $u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)$, *there exists* $T > 0$ *and a unique solution*

$$u \in C(I; \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2(I; \dot{\mathcal{N}}^{1-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2(I; \dot{B}^{-1}_{\infty 1}(\mathbb{R}^2))$$

to (1.2). *Besides* u *satisfies*

$$u \in C(\mathrm{Int}(I); \dot{\mathcal{N}}^{2-\delta}_{pq2}(\mathbb{R}^2)) \cap C^1(\mathrm{Int}(I); \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2))$$

and the flow map

$$u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2) \mapsto u \in L^\infty(I; \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2))$$

is Lipschitz continuous.

Theorem 1.3 (N.,Sawano)

Let $\kappa = \pm 1$. *Then there exists* $\varepsilon_0 > 0$ *such that for any* $u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^n)$ *with* $\|u_0\|_{\dot{\mathcal{N}}^{-\delta}_{pq2}} < \varepsilon_0$, *there exists a unique global solution*

$$u \in C([0,\infty), \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2))$$

to (1.2) *with*

$$u \in L^2((0,\infty), \dot{\mathcal{N}}^{1-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2((0,\infty); \dot{B}^{-1}_{\infty 1}(\mathbb{R}^2)) \cap C((0,\infty), \dot{\mathcal{N}}^{2-\delta}_{pq2}(\mathbb{R}^2)) \cap C^1((0,\infty), \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)).$$

Function spaces

For $f \in L^1(\mathbb{R}^n)$, define its Fourier transform and its inverse Fourier transform by

$$\mathcal{F}f(\xi) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\mathrm{i}x\cdot\xi} \mathrm{d}x, \quad \mathcal{F}^{-1}f(x) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi) \mathrm{e}^{\mathrm{i}x\cdot\xi} \mathrm{d}\xi.$$

By a well-known method, we can extend $\mathcal{F}, \mathcal{F}^{-1}$ naturally to the Schwartz distribution space $\mathcal{S}'(\mathbb{R}^n)$.

Function spaces

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfy

$$\chi_{B(4)\setminus B(2)} \leq \varphi \leq \chi_{B(8)\setminus B(\frac{3}{2})}.$$

Then define $\varphi_j \equiv \varphi(2^{-j}\cdot)$.

Definition 1.4

Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. We define

$$\|f\|_{\dot{B}^s_{pr}} \equiv \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \left\| \mathcal{F}^{-1}\left[\varphi_j \mathcal{F} f\right] \right\|_{L^p} \right)^r \right)^{\frac{1}{r}}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$. The (homogeneous) *Besov space* $\dot{B}^s_{pq}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ for which the norm $\|f\|_{\dot{B}^s_{pq}}$ is finite.

Remark 1.5

(1) *For* $s \in \mathbb{R}, 1 \le p \le \infty$ *and* $1 \le r_1 \le r_2 \le \infty$*,*

$$\dot{B}^s_{pr_1}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{pr_2}(\mathbb{R}^n).$$

Function spaces

Remark 1.5

(1) *For* $s \in \mathbb{R}, 1 \leq p \leq \infty$ *and* $1 \leq r_1 \leq r_2 \leq \infty$,

$$\dot{B}^s_{pr_1}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{pr_2}(\mathbb{R}^n).$$

(2) *For* $1 \leq p \leq \infty$,

$$\dot{B}^0_{p1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{p\infty}(\mathbb{R}^n).$$

Function spaces

Remark 1.5

(1) *For* $s \in \mathbb{R}, 1 \leq p \leq \infty$ *and* $1 \leq r_1 \leq r_2 \leq \infty$,

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$$\dot{B}^0_{p1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{p\infty}(\mathbb{R}^n).$$

(3)

$$\dot{B}^0_{p2}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \qquad (2 \leq p < \infty)$$

$$L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{p2}(\mathbb{R}^n) \qquad (1 < p \leq 2)$$

Function spaces

Remark 1.5

(1) *For* $s \in \mathbb{R}, 1 \le p \le \infty$ *and* $1 \le r_1 \le r_2 \le \infty$,

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$$\dot{B}^0_{p2}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \qquad (2 \le p < \infty)$$

$$L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{p2}(\mathbb{R}^n) \qquad (1 < p \le 2)$$

(4)

$$\dot{B}^0_{pp}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \qquad (1 \le p \le 2)$$

$$L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{pp}(\mathbb{R}^n) \qquad (2 \le p \le \infty)$$

Function spaces

The role of parameters of $\dot{B}^s_{pr}$

s : smoothness,

r : interpolation index, more smoothness after s determined,

p : integrability.

Proposition 1.6

Let $s \in \mathbb{R}$ *and* $1 \le p, r < \infty$. *Then*

$$\|f\|_{\dot{B}^s_{pr}} \sim \|\Delta f\|_{\dot{B}^{s-2}_{pr}}, \quad \|\partial_j f\|_{\dot{B}^{s-1}_{pr}} \lesssim \|f\|_{\dot{B}^s_{pr}}.$$

Classical results for Besov spaces

Theorem 1.7 (Ogawa,Shimizu (2010) [4])

Let $\kappa = \pm 1$. *Then for* $u_0 \in \dot{B}^0_{12}(\mathbb{R}^2)$, *there exists* $T > 0$ *and a unique solution*

$$u \in C([0,T), \dot{B}^0_{12}(\mathbb{R}^2)) \cap L^2((0,T), \dot{B}^0_{12}(\mathbb{R}^2))$$

to (1.1).
Besides the solution is belonging to

$$C([0,T), \dot{B}^0_{12}(\mathbb{R}^2)) \cap C((0,T), \dot{B}^2_{12}(\mathbb{R}^2)) \cap C^1((0,T), \dot{B}^0_{12}(\mathbb{R}^2))$$

and the flow map

$$\dot{B}^0_{12}(\mathbb{R}^2) \ni u_0 \to u \in C([0,T), \dot{B}^0_{12}(\mathbb{R}^2))$$

is Lipschitz continuous.

Classical results for Besov spaces

Theorem 1.8 (Ogawa,Shimizu (2010) [4])

Let $\kappa = \pm 1$*. Then for* $u_0 \in \dot{B}^0_{12}(\mathbb{R}^2)$*, there exists* $\varepsilon_0 > 0$ *such that for* $\|u_0\|_{\dot{B}^0_{12}} < \varepsilon_0$*, there exists a unique global solution* u *to (1.1) such that*

$$u \in C([0,\infty), \dot{B}^0_{12}(\mathbb{R}^2)) \cap L^2((0,\infty), \dot{B}^1_{12}(\mathbb{R}^2)) \cap C((0,\infty), \dot{B}^2_{12}(\mathbb{R}^2)) \cap C^1((0,\infty), \dot{B}^0_{12}(\mathbb{R}^2)).$$

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[1] T. Ogawa and S. Shimizu, End-point maximal regularity and its application to two-dimensional Keller-Segel system, Math. Z., 2010. **264**:601–628 DOI 10.1007/s00209-009-0481-3

Definition 1.9 (Morrey spaces)

Let $1 \le q \le p < \infty$. Define

$$\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} \equiv \sup \left\{ |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} : \ Q \text{ is a cube in } \mathbb{R}^n \right\}$$

for a measurable function f. The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all measurable functions f for which $\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$ is finite.

Main theorem 1(Well-posedness on Besov–Morrey spaces)

- For $1 \le q \le p < \infty$, Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is Banach space.
- $0 < q_1 < q_2 \le p < \infty$

$$L^p(\mathbb{R}^n) = \mathcal{M}^p_p(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_1}(\mathbb{R}^n).$$

- $|x|^{-\frac{n}{p}} \in \mathcal{M}^p_q(\mathbb{R}^n) \setminus L^p(\mathbb{R}^n)$ for $q < p$.
- Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is not reflexive nor separable.
- $C^\infty_c(\mathbb{R}^n)$ is not dense in $\mathcal{M}^p_q(\mathbb{R}^n)$.

Main theorem 1(Well-posedness on Besov–Morrey spaces)

Definition 1.10

Let $s \in \mathbb{R}$, $1 \le q \le p < \infty$ and $1 \le r \le \infty$. We define

$$\|f\|_{\dot{\mathcal{N}}^s_{pqr}} \equiv \left(\sum_{j=-\infty}^{\infty} \left(2^{js} \left\| \mathcal{F}^{-1} \left[\varphi_j \mathcal{F} f \right] \right\|_{\mathcal{M}^p_q} \right)^r \right)^{\frac{1}{r}}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$. The homogeneous *Besov–Morrey space* $\dot{\mathcal{N}}^s_{pqr}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ for which the norm $\|f\|_{\dot{\mathcal{N}}^s_{pqr}}$ is finite.

Remark 1.11

(1) *For* $1 \leq p < \infty$ *and* $s \in \mathbb{R}$

$$\dot{\mathcal{N}}^{s}_{ppr}(\mathbb{R}^n) = \dot{B}^{s}_{pr}(\mathbb{R}^n).$$

(2) *For* $1 \leq q_2 \leq q_1 \leq p < \infty$ *and* $1 \leq r_1 \leq r_2 \leq \infty$*,*

$$\dot{\mathcal{N}}^{s}_{pq_1r_1}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{N}}^{s}_{pq_2r_2}(\mathbb{R}^n).$$

(3) *(The role of parameters)*

s : *smoothness,*

r : *interpolation index, more smoothness after* s *determined,*

p : *global integrability,*

q : *local integrability.*

Main theorem 1(Well-posedness on Besov–Morrey spaces)

Remark 1.12

For $1 \le q \le p < \infty$,

$$H^1 \hookrightarrow \dot{B}^0_{12} = \dot{\mathcal{N}}^0_{112} \hookrightarrow \dot{\mathcal{N}}^{\frac{n}{p}-n}_{pp2} \hookrightarrow \dot{\mathcal{N}}^{\frac{n}{p}-n}_{pq2}.$$

Proposition 1.13

Let $1 \le q_1 \le p_1 < \infty$, $1 \le q_2 \le p_2 < \infty$, $-\infty < s_2 < s_1 < \infty$ *and* $1 \le r \le \infty$ *satisfy*

$$s = s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}, \quad \frac{q_1}{p_1} = \frac{q_2}{p_2}. \tag{1.3}$$

Then $\dot{\mathcal{N}}^{s_1}_{p_1 q_1 r}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{N}}^{s_2}_{p_2 q_2 r}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{\infty r}$.

Main theorem 1(Well-posedness on Besov–Morrey spaces)

For the scaling,

$$u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x),$$

$\dot{\mathcal{N}}^{\frac{2}{p}-2}_{pq2}(\mathbb{R}^2)$: scaling invariant function spaces

Main theorem 1(Well-posedness on Besov–Morrey spaces)

Theorem 1.2

Let $\kappa = \pm 1$. Let $1 \le q \le p < 2$. Define $\delta \equiv 2 - \dfrac{2}{p}$. Write $I = [0, T)$.

Then for $u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)$, there exists $T > 0$ and a unique solution

$$u \in C(I; \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2(I; \dot{\mathcal{N}}^{1-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2(I; \dot{B}^{-1}_{\infty 1}(\mathbb{R}^2))$$

to (1.2). Besides u satisfies

$$u \in C(\mathrm{Int}(I); \dot{\mathcal{N}}^{2-\delta}_{pq2}(\mathbb{R}^2)) \cap C^1(\mathrm{Int}(I); \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2))$$

and the flow map

$$u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2) \mapsto u \in L^\infty(I; \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2))$$

is Lipschitz continuous.

Important ingredients

$$\Phi(u) \equiv e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathrm{div}(u(s)\nabla\psi(s))\mathrm{d}s$$

- Maximal regularity
- Paraproducts
- Banach fixed point theorem

Lemma 1.14

Suppose that $1 \le q \le p < \infty, 1 \le r < \infty, s > 0$. *Then we have*

$$\|f \cdot g\|_{\dot{\mathcal{N}}^{s}_{pqr}} \lesssim \|f\|_{\dot{B}^{-1}_{\infty\infty}} \|g\|_{\dot{\mathcal{N}}^{s+1}_{pqr}} + \|f\|_{\dot{\mathcal{N}}^{s}_{pqr}} \|g\|_{\dot{B}^{0}_{\infty 1}}$$

for all $f \in \dot{B}^{-1}_{\infty\infty}(\mathbb{R}^n) \cap \dot{\mathcal{N}}^{s}_{pqr}(\mathbb{R}^n)$ *and* $g \in \dot{\mathcal{N}}^{s+1}_{pqr}(\mathbb{R}^n) \cap \dot{B}^{0}_{\infty 1}(\mathbb{R}^n)$.

Lemma 1.15

Let $1 \le q \le p < 2$ *and* $k \in \mathbb{N}$*. Write* $\delta \equiv 2 - \dfrac{n}{p}$*. Then we have*

$$\|v\nabla(-\Delta)^{-1}w\|_{\dot{\mathcal{N}}^{k-\delta}_{pq2}} \lesssim \|v\|_{\dot{\mathcal{N}}^{1-\delta}_{pq2}}\|w\|_{\dot{\mathcal{N}}^{k-\delta}_{pq2}} + \|v\|_{\dot{\mathcal{N}}^{k-\delta}_{pq2}}\|w\|_{\dot{B}^{-1}_{\infty 1}}$$

for all $v \in \dot{\mathcal{N}}^{1-\delta}_{pq2}(\mathbb{R}^n) \cap \dot{\mathcal{N}}^{k-\delta}_{pq2}(\mathbb{R}^n)$ *and* $w \in \dot{\mathcal{N}}^{k-\delta}_{pq2}(\mathbb{R}^n) \cap \dot{B}^{-1}_{\infty 1}(\mathbb{R}^n)$*.*

Contents

Maximal regularity

We consider the following initial value problem for the abstract evolution equation

$$\frac{du}{dt} + Au = f \quad (t > 0), \quad u(0) = 0. \tag{2.1}$$

Here,

- $0 < t \leq T \leq \infty, f \in L^p((0,T),X)$ $(1 < p < \infty)$
- A : closed linear operator densely defined on X

Maximal regularity

Then, A has maximal L^p-regularity $\Longleftrightarrow$

the equation (2.1) has a solution $u \in W^{1,p}((0,T),X) \cap L^p((0,T),D(A))$ and the following estimate holds:

$$\left\| \frac{du}{dt} \right\|_{L^p((0,T),X)} + \|Au\|_{L^p((0,T),X)} \leq C\|f\|_{L^p((0,T),X)}. \tag{2.2}$$

Maximal regularity

- Ladyzhenskaya-Solonnikov-Ural'tseva (1964)
- De Simon (1964) : Hilbert space
- Sobolevskii (1975) : Banach space
- Da Prato-Grisvard (1975) : operator sum method
- Dore-Venni (1987) : UMD (unconditional martingale differences)

Maximal regularity

- Ladyzhenskaya-Solonnikov-Ural'tseva (1964)
- De Simon (1964) : Hilbert space
- Sobolevskii (1975) : Banach space
- Da Prato-Grisvard (1975) : operator sum method
- Dore-Venni (1987) : UMD (unconditional martingale differences)

Definition 2.1

A Banach space X called UMD (unconditional martingale differences) space if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for $1 < p < \infty$.

Background

Weis (2001)

the operator A on UMD Banach space has L^p maximal regularity

$\Updownarrow$ iff

$\mathcal{R}$-boundedness for the resolvent of A

$\updownarrow$

boundedness of the operator-valued Fourier multiplier

$\Longrightarrow$ The property "UMD" is important!

Maximal regularity

Proposition 2.2

UMD *Banach space is reflexive. However, the opposite is not true.*

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- Banach space without the property "UMD" (for example L^1, L^∞, H^1) must be considered for each case.

Maximal regularity

Proposition 2.2

UMD *Banach space is reflexive. However, the opposite is not true.*

- Banach space without the property "UMD" (for example L^1, L^∞, H^1) must be considered for each case.
- The property "UMD" is not nessesary condition to hold maximal reguarity.
- homogeneous Besov space $\dot{B}^0_{1p}$ (Ogawa, Shimizu [4])

Main theorem 2 (Maximal regularity in Besov–Morrey spaces)

We consider the heat equation

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^{n+1}_+, \\ u(0,\cdot) = u_0 & \text{on } \mathbb{R}^n. \end{cases} \tag{2.3}$$

Main theorem 2 (Maximal regularity in Besov–Morrey spaces)

Ogawa and Shimizu proved the maximal regularity for the heat equation in Besov spaces $\dot{B}^0_{1q}$ which is not reflexive. ([4][2])

Theorem 2.3 (Ogawa, Shimizu (2010))

Let $1 < q \leq \infty$ *and* $I = [0, T)$ *be an interval with* $T \leq \infty$. *For* $f \in L^q(I, \dot{B}^0_{1q}(\mathbb{R}^n))$ *and* $u_0 \in \dot{B}^{2(1-1/q)}_{1q}(\mathbb{R}^n)$, *let* u *be a solution of the heat equation (2.3). Then, we have*

$$\left\| \frac{du}{dt} \right\|_{L^q(I,\dot{B}^0_{1q})} + \|\nabla^2 u\|_{L^q(I,\dot{B}^0_{1q})} \leq C \left(\|u_0\|_{\dot{B}^{2(1-1/q)}_{1q}} + \|f\|_{L^q(I,\dot{B}^0_{1q})} \right), \tag{2.4}$$

where $\nabla^2 = \partial_{x_j}\partial_{x_k}$ *for all* $1 \leq j, k \leq n$.

[2] T. Ogawa and S. Shimizu, End-point maximal regularity and its application to two-dimensional Keller-Segel system, Math. Z., 2010. **264**:601–628 DOI 10.1007/s00209-009-0481-3

Main theorem 2 (Maximal regularity in Besov–Morrey spaces)

Theorem 2.4 (N., Sawano)

Let $1 \le q \le p < \infty$, $1 \le \rho \le \infty$. *Consider the heat equation* (2.3) *with* $f \in L^\rho([0,\infty); \dot{\mathcal{N}}^0_{pq\rho}(\mathbb{R}^n))$ *and* $u_0 \in \dot{\mathcal{N}}^{2-2/\rho}_{pq\rho}(\mathbb{R}^n)$. *Then*

$$\|\partial_t u\|_{L^\rho([0,\infty);\dot{\mathcal{N}}^0_{pq\rho})} + \|\nabla^2 u\|_{L^\rho([0,\infty);\dot{\mathcal{N}}^0_{pq\rho})} \lesssim \|u_0\|_{\dot{\mathcal{N}}^{2-2/\rho}_{pq\rho}} + \|f\|_{L^\rho([0,\infty);\dot{\mathcal{N}}^0_{pq\rho})}.$$

Main theorem 2 (Maximal regularity in Besov–Morrey spaces)

Proposition 2.5

(1) *Let* $1 \le q \le p < \infty$ *and* $1 \le \rho < \infty$. *Then*

$$\left(\int_0^\infty (\|\nabla \exp(t\Delta)u_0\|_{\dot{\mathcal{N}}^0_{pq1}})^\rho \mathrm{d}t\right)^{\frac{1}{\rho}} \lesssim \|u_0\|_{\dot{\mathcal{N}}^{1-\frac{2}{\rho}}_{pq\rho}}$$

for all $u_0 \in \dot{\mathcal{N}}^{1-\frac{2}{\rho}}_{pq\rho}(\mathbb{R}^n)$.

(2) *Let* $1 \le q \le p < \infty$ *and* $1 \le \rho \le \infty$. *Then*

$$\sup_{t>0} \|\nabla \exp(t\Delta)u_0\|_{\dot{\mathcal{N}}^0_{pq\rho}} \lesssim \|u_0\|_{\dot{\mathcal{N}}^1_{pq\rho}}$$

for all $u_0 \in \dot{\mathcal{N}}^1_{pq\rho}(\mathbb{R}^n)$.

Main theorem 2 (Maximal regularity in Besov–Morrey spaces)

Proposition 2.5 (cont.)

(3) For all $u_0 \in \dot{B}^0_{\infty 2}(\mathbb{R}^n)$,

$$\left(\int_0^\infty (\|\nabla \exp(t\Delta) u_0\|_{\dot{B}^0_{\infty 1}})^2 \mathrm{d}t \right)^{\frac{1}{2}} \lesssim \|u_0\|_{\dot{B}^0_{\infty 2}}.$$

Main theorem 2 (Maximal regularity in Besov–Morrey spaces)

Proposition 2.6

Let $1 \le q \le p < \infty$ *and* $1 \le \rho \le \infty$. *Then*

$$\left\| \nabla \int_0^t \exp((t-s)\Delta) f(s) \mathrm{d}s \right\|_{L^\rho(0,\infty;\dot{\mathcal{N}}^0_{pq\rho})} \lesssim \|f\|_{L^\rho(0,\infty;\dot{\mathcal{N}}^{-1}_{pq\rho})}$$

for all $f \in L^\rho(0,\infty;\dot{\mathcal{N}}^{-1}_{pq\rho}(\mathbb{R}^n))$.

Main theorem 2 (Maximal regularity in Besov–Morrey spaces)

Corollary 2.7

Let $1 \le q \le p < \infty$ *and* $1 \le \rho \le \infty$. *Then we have*

$$\left\| \Delta \int_0^t e^{(t-s)\Delta} f(s) \mathrm{d}s \right\|_{L^\rho([0,\infty);\dot{\mathcal{N}}^0_{pq\rho})} \lesssim \|f\|_{L^\rho([0,\infty);\dot{\mathcal{N}}^0_{pq\rho})}$$

for all $f \in L^\rho([0,\infty);\dot{\mathcal{N}}^0_{pq\rho}(\mathbb{R}^n))$.

Main theorem 2 (Maximal regularity in Besov–Morrey spaces)

We will use this estimate for the proof of the local existence of solution of the equation (1.1).

Proposition 2.8

Let $1 \le q \le p < \infty$, $1 \le \sigma \le \infty$.
(1) *For all* $f \in L^1([0,\infty); \dot{\mathcal{N}}^0_{pq\sigma}(\mathbb{R}^n))$,

$$\left\| \int_0^t \exp((t-s)\Delta) f(s) \mathrm{d}s \right\|_{L^\infty([0,\infty); \dot{\mathcal{N}}^0_{pq\sigma})} \lesssim \|f\|_{L^1([0,\infty); \dot{\mathcal{N}}^0_{pq\sigma})}.$$

Main theorem 2 (Maximal regularity in Besov–Morrey spaces)

Proposition 2.8 (cont.)

(2) For all $f \in L^1([0,\infty); \dot{\mathcal{N}}^0_{pq2}(\mathbb{R}^n))$,

$$\left\| \int_0^t \nabla[\exp((t-s)\Delta)f(s)]\mathrm{d}s \right\|_{L^2([0,\infty);\dot{\mathcal{N}}^0_{pq1})} \lesssim \|f\|_{L^1([0,\infty);\dot{\mathcal{N}}^0_{pq2})}.$$

(3) For all $f \in L^1([0,\infty); \dot{B}^0_{\infty 2}(\mathbb{R}^n))$,

$$\left\| \int_0^t \nabla[\exp((t-s)\Delta)f(s)]\mathrm{d}s \right\|_{L^2([0,\infty);\dot{B}^0_{\infty 1})} \lesssim \|f\|_{L^1([0,\infty);\dot{B}^0_{\infty 2})}.$$

Contents

Sketch of the proofs

Ogawa, Shimizu [4]

- real interpolation
- duality

$\Longrightarrow$ However, these ingredients don't work well in Morrey spaces.

$\Longrightarrow$ We use smoothness and vanishing moments.

Sketch of the proofs

Proposition 2.5

$$(1) \quad \left(\int_0^\infty (\|\nabla \exp(t\Delta)u_0\|_{\dot{\mathcal{N}}^0_{pq1}})^\rho \mathrm{d}t\right)^{\frac{1}{\rho}} \lesssim \|u_0\|_{\dot{\mathcal{N}}^{1-\frac{2}{\rho}}_{pq\rho}}$$

Let $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ satisfy

$$\chi_{B(4)\setminus B(2)} \leq \varphi \leq \chi_{B(8)\setminus B(\frac{3}{2})}.$$

Let $\Phi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ be an even function that vanishes on $|x| \leq 1$ and assumes 1 on the support of φ.

Sketch of the proofs

Then, we have

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \|\nabla\varphi_j(D)\exp(t\Delta)u_0\|_{\mathcal{M}_q^p} &= \sum_{j=-\infty}^{\infty} \|\nabla\Phi_j(D)\exp(t\Delta)\varphi_j(D)u_0\|_{\mathcal{M}_q^p} \\
&\lesssim \sum_{j=-\infty}^{\infty} 2^j \|\exp(t\Delta)\varphi_j(D)u_0\|_{\mathcal{M}_q^p} \\
&= \sum_{j=-\infty}^{\infty} 2^j \|\Phi_j(D)\exp(t\Delta)\varphi_j(D)u_0\|_{\mathcal{M}_q^p} \\
&\lesssim \sum_{j=-\infty}^{\infty} 2^j \exp(-4^j t) \left\|\mathcal{F}^{-1}\left[\varphi_j \mathcal{F}u_0\right]\right\|_{\mathcal{M}_q^p}.
\end{aligned}$$

Sketch of the proofs

(Proof of the local well-posedness)

$$X_T \equiv X_T(I) = \left\{ h \in C(I; \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)) \,:\, \|h\|_{L^\infty(I;\dot{\mathcal{N}}^{-\delta}_{pq2})} \leq \frac{3}{2} M_0 \right\}$$
$$\cap \left\{ h \in L^2(I; \dot{\mathcal{N}}^{1-\delta}_{pq2}(\mathbb{R}^2) \cap \dot{B}^{-1}_{\infty 1}(\mathbb{R}^2)) \,: \right.$$
$$\left. \|\nabla h\|_{L^2(I;\dot{\mathcal{N}}^{-\delta}_{pq2})} + \|\nabla h\|_{L^2(I;\dot{B}^{-2}_{\infty 1})} \leq \frac{3}{2} M \right\}$$

$$\|h\|_{X_T} = \|h\|_{L^\infty(I;\dot{\mathcal{N}}^{-\delta}_{pq2})} + \|\nabla h\|_{L^2(I;\dot{\mathcal{N}}^{-\delta}_{pq2})} + \|\nabla h\|_{L^2(I;\dot{B}^{-2}_{\infty 1})}$$

for $h \in L^\infty(I; \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2(I; \dot{\mathcal{N}}^{1-\delta}_{pq2}(\mathbb{R}^2)) \cap L^2(I; \dot{B}^{-1}_{\infty 1}(\mathbb{R}^2))$.

Sketch of the proofs

$$\Xi(u)(t) \equiv e^{t\Delta}u_0 - \kappa \int_0^t e^{(t-s)\Delta} \left[\mathrm{div}[u(s)\nabla(-\Delta)^{-1}u(s)]\right] \mathrm{d}s.$$

Proposition 3.1

Let $u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)$*, and let* $0 < T, M \ll 1$*. Then* Ξ *mapsto* X_T *to itself.*

Proposition 3.2

Let $u_0 \in \dot{\mathcal{N}}^{-\delta}_{pq2}(\mathbb{R}^2)$*. Then*

$$\|\Xi(u_1) - \Xi(u_2)\|_{X_T} \leq \frac{1}{4}\|u_1 - u_2\|_{X_T} \quad (u_1, u_2 \in X_T)$$

as long as T *and* M *are small enough.*

Thank you for your attentions!

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