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On a class of GJMS equations on the standard n-sphere

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Outline of the talk

Review on functional inequalities of Sobolev type detail

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A perturbation approach by Hang and Yang detail

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Main result detail

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Technical difficulty detail

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Discussion detail

Second order Sobolev's inequality for $\mathbf{R}^n$: for $n \geq 3$ and $1 < p \leq \frac{2n}{n-2}$, we have

$$\left(\int_{\mathbf{R}^n} |u|^p dx \right)^{2/p} \leq K_{n,p} \int_{\mathbf{R}^n} |\nabla u|^2 dx \tag{1}$$

for all $u \in W^{1,2}(\mathbf{R}^n)$.

[Sharp form of (1) was independently found by Aubin and Talenti in 1976.]

[Inequality (1) can be thought of as the continuity of the embedding $W^{1,2}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$ up to $p = \frac{2n}{n-2}$.]

Second order Sobolev's inequality for $(\mathbb{S}^n, g_{\mathbb{S}^n})$: on the standard sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$

$$\left(\int_{\mathbb{S}^n} |v|^p d\mu_{\mathbb{S}^n} \right)^{2/p} \leq \frac{p-2}{n} \int_{\mathbb{S}^n} |\nabla v|^2 d\mu_{\mathbb{S}^n} + \int_{\mathbb{S}^n} |v|^2 d\mu_{\mathbb{S}^n} \tag{2}$$

for $n \geq 3$, $2 < p \leq \frac{2n}{n-2}$, and all $v \in W^{1,2}(\mathbb{S}^n)$.

[Sharp form of (2) was proved by Beckner in 1993 using spherical harmonics and the dual-spectral form of the Hardy–Littlewood–Sobolev inequality on $\mathbb{S}^n$.]

[Inequality (2) can also be obtained directly from (1) by making use of stereographic projection.]

[The case $p = \frac{2n}{n-2}$ is of particularly interesting.]

Critical Sobolev's inequality for $(\mathbb{S}^n, g_{\mathbb{S}^n})$: With $2 < p \leq \frac{2n}{n-2}$, recall from (2)

$$\Big(\int_{\mathbb{S}^n} |v|^p d\mu_{\mathbb{S}^n}\Big)^{2/p} \leq \frac{p-2}{n}\int_{\mathbb{S}^n} |\nabla v|^2 d\mu_{\mathbb{S}^n} + \int_{\mathbb{S}^n} |v|^2 d\mu_{\mathbb{S}^n}.$$

In the critical case $p = \frac{2n}{n-2}$ with $n \geq 3$ the critical Sobolev inequality is

$$\Big(\int_{\mathbb{S}^n} |v|^{\frac{2n}{n-2}} d\mu_{\mathbb{S}^n}\Big)^{\frac{n-2}{n}} \leq \frac{4}{n(n-2)}\int_{\mathbb{S}^n} |\nabla v|^2 d\mu_{\mathbb{S}^n} + \int_{\mathbb{S}^n} |v|^2 d\mu_{\mathbb{S}^n}.$$

If we denote

$$\mathbf{L}_n^2 : v \mapsto -\Delta v + \frac{n(n-2)}{4} v,$$

then the critical Sobolev inequality can be rewritten as

$$\Big(\int_{\mathbb{S}^n} |v|^{\frac{2n}{n-2}} d\mu_{\mathbb{S}^n}\Big)^{\frac{n-2}{n}} \leq \frac{4}{n(n-2)}\int_{\mathbb{S}^n} v\mathbf{L}_n^2(v) d\mu_{\mathbb{S}^n} \tag{3}$$

[This is because

$$\int_{\mathbb{S}^n} v\mathbf{L}_n^2(v) d\mu_{\mathbb{S}^n} = \int_{\mathbb{S}^n} v\big(-\Delta v + \frac{n(n-2)}{4} v\big) d\mu_{\mathbb{S}^n} = \cdots .]$$

[$\mathbf{L}_n^2$ is known as the conformal Laplacian on $\mathbb{S}^n$, which is of second order. And we are interested in cases of higher order operators instead of second order operator $\mathbf{L}_n^2$.]

The conformal Laplacian of second-order on $\mathbb{S}^n$

$$\mathbf{L}_n^2 = -\Delta + \frac{n(n-2)}{4}$$

is an example of lower-order conformal transformations. The first example of higher-order conformal transformations was found by Paneitz in 1983.

On $(\mathbb{S}^n, g_{\mathbb{S}^n})$ with $n \geq 3$, this operator, denoted by $\mathbf{P}_n^4$ and called Paneitz's operator, is as follows

$$\mathbf{P}_n^4 = \big(-\Delta + \frac{n(n-2)}{4}\big)\big(-\Delta + \frac{(n+2)(n-4)}{4}\big)$$

The other example of higher-order conformal transformations was found by Graham, Jenne, Mason, and Sparling in 1992.

On $(\mathbb{S}^n, g_{\mathbb{S}^n})$ with $n \geq 3$, this operator of order $2m$, denoted by $\mathbf{P}_n^{2m}$ and called GJMS's operator, is as follows

$$\mathbf{P}_n^{2m} = \prod_{i=0}^{m-1} \big(-\Delta + \frac{(n+2i)(n-2i-2)}{4}\big)$$

[In general, to define $\mathbf{P}_n^{2m}$ it is required either $3 \leq n$ is odd or $2m \leq n$ is even.]

Recall (3), that is

$$\Big(\int_{\mathbb{S}^n} |v|^{\frac{2n}{n-2}} d\mu_{\mathbb{S}^n}\Big)^{\frac{n-2}{n}} \leq \frac{4}{n(n-2)} \int_{\mathbb{S}^n} v\mathbf{L}_n^2(v) d\mu_{\mathbb{S}^n}.$$

A natural generalization of (3) for Paneitz's operator could be

$$\Big(\int_{\mathbb{S}^n} |v|^p d\mu_{\mathbb{S}^n}\Big)^{2/p} \lesssim_{n,p} \int_{\mathbb{S}^n} v\mathbf{P}_n^4(v) d\mu_{\mathbb{S}^n} \tag{4}$$

for $1 < p \leq \frac{2n}{n-4}$ if $n \geq 5$ and $-6 = \frac{2\cdot 3}{3-4} \leq p < 0$ if $n = 3$.

[The 4th order Sobolev's inequality for $\mathbf{R}^n$: for $n \geq 5$ and $1 < p \leq \frac{2n}{n-4}$, we have

$$\Big(\int_{\mathbf{R}^n} |u|^p dx\Big)^{2/p} \lesssim_{n,p} \int_{\mathbf{R}^n} (\Delta u)^2 dx \quad \forall u \in W^{2,2}(\mathbf{R}^n).] \tag{5}$$

[It appears that in (4) the two cases $n < 4$ and $n \geq 5$ could be very different.]

[In the case $n = 3$, as $\mathbf{P}_3^4(1) = -15/16 < 0$, the RHS of (4) is strictly negative.]

Similarly, a natural generalization of (3) for GJMS's operator could be

$$\Big(\int_{\mathbb{S}^n} |v|^p d\mu_{\mathbb{S}^n}\Big)^{2/p} \lesssim_{n,p} \int_{\mathbb{S}^n} v\mathbf{P}_n^{2m}(v) d\mu_{\mathbb{S}^n} \tag{6}$$

for $1 < p \leq \frac{2n}{n-2m}$ if $n > 2m$ and $\frac{2n}{n-2m} \leq p < 0$ if $3 \leq n < 2m$.

[It appears that in (4) the two cases $n < 2m$ and $n > 2m$ could also be very different.]

In 1993 (a preprint appeared in 1991), Beckner proved (6) for $n > 2m \geq 4$

$$\left(\int_{\mathbb{S}^n} |v|^p d\mu_{\mathbb{S}^n} \right)^{2/p} \lesssim_{n,p} \int_{\mathbb{S}^n} v\mathbf{P}_n^{2m}(v) d\mu_{\mathbb{S}^n}$$

[The method used is based on spherical harmonics.]

[The above inequality also includes (4) for all $n \geq 5$.]

In 2004 (a preprint appeared in 2003), Yang and Zhu proved (4) in the critical case in the remaining case $n = 3$

$$\left(\int_{\mathbb{S}^3} |v|^{-6} d\mu_{\mathbb{S}^n} \right)^{-3} \lesssim \int_{\mathbb{S}^3} v\mathbf{P}_3^4(v) d\mu_{\mathbb{S}^3} \qquad (7)$$

[If $n = 3$, then $\frac{2n}{n-4} = -6$. The method used is based on symmetrization.]

In 2007 (a preprint appeared in 2003), Zhu proved (8) in the critical case for odd $n \in \{3, ..., 2m\}$

$$\left(\int_{\mathbb{S}^n} |v|^{-\frac{2n}{2m-n}} d\mu_{\mathbb{S}^n} \right)^{-\frac{2m-n}{n}} \lesssim_{n,p} \int_{\mathbb{S}^n} v\mathbf{P}_n^{2m}(v) d\mu_{\mathbb{S}^n} \qquad (8)$$

[In fact, the RHS of (8) is bounded from below if either $n = 2m - 1$ or $n = 2m - 3$. The method used is variational.]

In 2018, F. Hang and P. Yang (arXiv:1802.09692) proposed the following alternative approach to prove (7). For small $\varepsilon > 0$, consider

$$\inf_{0<\phi\in W^{2,2}(\mathbb{S}^3)} \Big(\int_{\mathbb{S}^3} |\phi|^{-6} d\mu_{\mathbb{S}^n} \Big)^3 \int_{\mathbb{S}^3} \phi \big[\mathbf{P}_3^4(\phi) + \varepsilon\phi \big] d\mu_{\mathbb{S}^3} \tag{9}$$

It can be proved that there is a smooth minimizer $v_\varepsilon > 0$ to (9). In addition, v_ε solves

$$\mathbf{P}_3^4(v_\varepsilon) + \varepsilon v_\varepsilon = -v_\varepsilon^{-7} \quad \text{on } \mathbb{S}^3$$

If any smooth, positive solution to the above PDE is constant, then for any $\phi \in W^{2,2}(\mathbb{S}^3)$

$$\Big(\int_{\mathbb{S}^3} |\phi|^{-6} d\mu_{\mathbb{S}^n} \Big)^3 \int_{\mathbb{S}^3} \phi \big[\mathbf{P}_3^4(\phi) + \varepsilon\phi \big] d\mu_{\mathbb{S}^3} \geq \big[\mathbf{P}_3^4(1) + \varepsilon \big] |\mathbb{S}^3|^{4/3}$$

Letting $\varepsilon \searrow 0$ gives the desired inequality with a sharp constant

$$-\frac{15}{16} |\mathbb{S}^3|^{4/3}$$

[Recall that $\mathbf{P}_3^4 = (-\Delta + \frac{3}{4})(-\Delta - \frac{5}{4}) = \Delta^2 + \frac{1}{2}\Delta - \frac{15}{16}$.]

[Hang and Yang proposed this approach, but in their paper, they used a way around.]

For small $\varepsilon > 0$, recall if u_ε is a minimizer to (9), namely

$$\inf_{0<\phi\in W^{2,2}(\mathbb{S}^3)} \Big(\int_{\mathbb{S}^3} |\phi|^{-6} d\mu_{\mathbb{S}^n} \Big)^3 \int_{\mathbb{S}^3} \phi \big[\mathbf{P}_3^4(\phi) + \varepsilon\phi \big] d\mu_{\mathbb{S}^3},$$

then up to a constant multiple u_ε solves

$$\mathbf{P}_3^4(v) + \varepsilon v = -v^{-7} \quad \text{on } \mathbb{S}^3. \tag{10}$$

Hang and Yang raised the following:

Conjecture

Let $\varepsilon > 0$ be a small number. If v is a positive, smooth function solution to (10), then v must be a constant function.

[In their work, Hang and Yang worked on minimizers. Being a minimizer, there is an extra freedom, namely one assumes

$$\int_{\mathbb{S}^3} |\phi|^{-6} d\mu_{\mathbb{S}^n} = 1,$$

which is not available for any solution to the PDE. Hence, the conjecture is about to ask for a larger class of optimizers without any constraint.]

This conjecture was recently confirmed by Shihong Zhang (arXiv:2104.03060).

Inspired by Hang and Yang's conjecture and the work of Zhang, we aim to study Liouville type result for

$$\mathbf{P}_n^{2m}(v) = \underbrace{Q_n^{2m}}_{\mathbf{P}_n^{2m}(1)} (\varepsilon v + v^{-\alpha}) \quad \text{on } \mathbb{S}^n \tag{11}$$

under

$$3 \le n < 2m, \quad n \text{ is odd}, \quad \alpha > 0, \quad \varepsilon \in [0,1)$$

Here recall $\mathbf{P}_n^{2m}$ is GJMS's operator of order $2m$ on $\mathbb{S}^n$ with

$$\mathbf{P}_n^{2m} = (-\Delta)^m + \text{l.o.t} + Q_n^{2m}$$

[Q_n^{2m} does not have a sign, for example $Q_3^4 < 0$ but $Q_3^6 > 0$. Fortunately, $Q_n^{2m} \neq 0$.]
[Now the condition $\varepsilon \in [0,1)$ can be easily seen by integrating both sides of (11) to get

$$(1-\varepsilon) \int_{\mathbb{S}^n} v \, d\mu_{g_{\mathbb{S}^n}} = \int_{\mathbb{S}^n} v^{-\alpha} d\mu_{g_{\mathbb{S}^n}}$$

after canceling both sides by Q_n^{2m}.]

Our aim is to show that for suitable small $\varepsilon \in (0,1)$ and $0 < \alpha \le \frac{2m+n}{2m-n}$, any smooth, positive solution to (11) must be constant.

back to Hang–Yang's conjecture

However, we need to modify Zhang's approach.

to our approach

Our main result reads as follows:

Theorem (the negative case, namely $n < 2m$)

Let assume $n \geq 3$ be odd and $m > n/2$. Then there exists $\varepsilon_* \in (0,1)$ such that under one of the following conditions

1. *either $\varepsilon \in (0, \varepsilon_*)$ and*
$$0 < \alpha \leq \frac{2m+n}{2m-n}$$
2. *or $\varepsilon = 0$ and*
$$0 < \alpha < \frac{2m+n}{2m-n}$$

any positive, smooth solution v to

$$\mathbf{P}_n^{2m}(v) = Q_n^{2m}(\varepsilon v + v^{-\alpha}) \quad \text{on } \mathbb{S}^n$$

must be constant, hence is equal to $(1-\varepsilon)^{-1/(\alpha+1)}$.

Next, let us briefly sketch our approach. It consists of three main steps as follows: (1) to derive some integral equation for u on $\mathbf{R}^n$, (2) to prove that u must be radially symmetric, and (3) to prove that v must be constant.

to positive case

Zhang used the following approach to tackle the conjecture:

$$\mathbf{P}_3^4(v) = -(\varepsilon v + v^{-7}) \quad \text{on } \mathbb{S}^3$$

$\downarrow$ via the stereographic projection

$$\Delta^2 u = -\Big[\varepsilon\Big(\frac{2}{1+|x|^2}\Big)^4 u + u^{-7}\Big] \quad \text{in } \mathbf{R}^3$$

$$u(x) = \frac{1}{8\pi}\int_{\mathbf{R}^3} |x-y|\Big[\varepsilon\Big(\frac{2}{1+|x|^2}\Big)^4 u(y) + u(y)^{-7}\Big]dy \quad \text{in } \mathbf{R}^3$$

$\downarrow$ via the Kelvin transform

$$\Delta^2 u^* = -\Big[\varepsilon\Big(\frac{2}{1+|x|^2}\Big)^4 u^* + (u^*)^{-7}\Big] \quad \text{in } \mathbf{R}^3 \setminus \{0\}$$

$\downarrow$ via the method of moving planes

u^* is radially symmetric and increasing

$\downarrow$

v is radially symmetric w.r.t. any critical point

$\downarrow$ via Kazdan–Warner type identity

v is constant

to our approach

We modify Zhang's approach to tackle the higher dimensional problem as follows:

$$\mathbf{P}_n^{2m}(v) = Q_n^{2m}(\varepsilon v + v^{-\alpha}) \text{ on } \mathbb{S}^n$$

$\downarrow$ via the stereographic projection

$$(-\Delta)^m u = Q_n^{2m}\Big[\varepsilon\Big(\frac{2}{1+|x|^2}\Big)^{2m} u + \Big(\frac{2}{1+|x|^2}\Big)^{\frac{n+2m}{2}+\alpha\frac{n-2m}{2}} u^{-\alpha}\Big] \text{ in } \mathbf{R}^n$$

$$u(x) = C\int_{\mathbf{R}^n} |x-y|^{2m-n}\Big[\varepsilon\Big(\frac{2}{1+|x|^2}\Big)^{2m} u + \Big(\frac{2}{1+|x|^2}\Big)^{\frac{n+2m}{2}+\alpha\frac{n-2m}{2}} u^{-\alpha}\Big] dy \text{ in } \mathbf{R}^n$$

$\downarrow$ via the Kelvin transform

$$(-\Delta)^m u^* = Q_n^{2m}\Big[\varepsilon\Big(\frac{2}{1+|x|^2}\Big)^{2m} u^* + \Big(\frac{2}{1+|x|^2}\Big)^{\frac{n+2m}{2}+\alpha\frac{n-2m}{2}} (u^*)^{-\alpha}\Big] \text{ in } \mathbf{R}^n$$

$\downarrow$ via the method of moving planes

u^* is radially symmetric and increasing

u is radially symmetric and increasing

$\downarrow$

v is radially symmetric w.r.t. any critical point

$\downarrow$ via Kazdan–Warner type identity

v is constant

to difficulties

There are at least three difficulties that we are going to describe.

The first difficulity is how to transfer

$$\mathbf{P}_n^{2m}(v) = Q_n^{2m}(\varepsilon v + v^{-\alpha}) \text{ on } \mathbb{S}^n$$

$$\downarrow \qquad u = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2m}{2}}(v \circ \pi^{-1})$$

$$u(x) = C\int_{\mathbf{R}^n} |x-y|^{2m-n} F_{\varepsilon,u}(y)dy \quad \text{in } \mathbf{R}^n$$

for some $F_{\varepsilon,u}$. Roughly speaking, there are at least two routes achieving this.

1. to exploit the sign of $(-\Delta)^i u$, the sub/super poly-harmonicity
2. to exploit the sign of $\int_{\mathbf{R}^n} u(-\Delta)^i\varphi$, the weakly sub/super poly-harmonicity.

Zhang essentially follows the first route by making use of techniques from potential analysis, which makes the analysis quite involved.

[In the published version, this part consists of nearly 10 pages.]

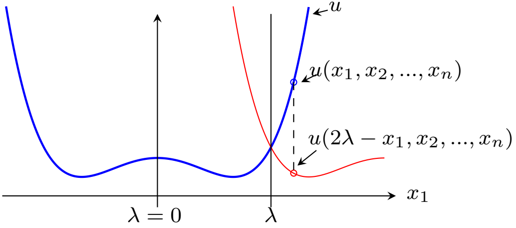
We offer a completely new approach by exploiting the relation of stereographic projections centered at different points.

Now let see why a compactness result is required. This is the second difficulty. Now we forget v on $\mathbb{S}^n$ but focus on u on $\mathbf{R}^n$. The aim is to prove u (blue curve in the figure below) is symmetric w.r.t. $x_1 = 0$. As

$$u(x) = \underbrace{\Big(\frac{2}{1+|x|^2}\Big)^{\frac{n-2m}{2}}}_{\nearrow +\infty} \underbrace{(v \circ \pi^{-1})(x)}_{\to v(\text{north pole})} \quad \text{as } |x| \nearrow +\infty$$

So for large $\lambda \gg 1$ and large $x_1 \gg \lambda$, one should have

$$u(x_1, x_2, ..., x_n) \geq u(2\lambda - x_1, x_2, ..., x_n) \quad \forall x_1 \gg \lambda$$



Then we lower $\lambda \searrow 0$ so long as $u(x_1, x_2, ...) \geq u(2\lambda - x_1, x_2, ...)$ remains valid. The key step is to show $\lambda = 0$. (Then we let $\lambda \nearrow 0$... to get symmetry.)

Denote $x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$ with $|x^\lambda| < |x|$ in $\{x_1 > \lambda > 0\}$. Recall

$$u(x) - u(x^\lambda) = C \int_{\{x_1>\lambda\}} \big[\underbrace{|x^\lambda - y|^{2m-n} - |x-y|^{2m-n}}_{\geq 0} \big] \big[F_\varepsilon(y^\lambda) - F_\varepsilon(y) \big] dy$$

with

$$F_\varepsilon(z) = \varepsilon \Big(\frac{2}{1+|z|^2} \Big)^{2m} u(z) + \Big(\frac{2}{1+|z|^2} \Big)^{\frac{n+2m}{2} + \alpha \frac{n-2m}{2}} u(z)^{-\alpha}.$$

To lower $\lambda \searrow 0$ one needs $u(x) > u(x^\lambda)$. And to gain $u(x) > u(x^\lambda)$, one wishes

$$F_\varepsilon(y^\lambda) - F_\varepsilon(y) \geq 0 \quad \forall y_1 > \lambda > 0.$$

As $F_\varepsilon(z)$ has two power terms with opposite sign: while in $\{x_1 > \lambda > 0\}$

$$\varepsilon \Big(\frac{2}{1+|y^\lambda|^2} \Big)^{\frac{n+2m}{2} + \alpha \frac{n-2m}{2}} u(y^\lambda)^{-\alpha} > \varepsilon \Big(\frac{2}{1+|y|^2} \Big)^{\frac{n+2m}{2} + \alpha \frac{n-2m}{2}} u(y)^{-\alpha},$$

which is good, the first term is not that good because one cannot claim

$$\varepsilon \Big(\frac{2}{1+|y^\lambda|^2} \Big)^{2m} u(y^\lambda) \overset{???}{\geq} \varepsilon \Big(\frac{2}{1+|y|^2} \Big)^{2m} u(y).$$

This requires some control of u independent of ε, leading to a compactness result. We make use of this compactness result as follows

$$u(y) \ll u(y)^{-\alpha} \ll u(y^\lambda)^{-\alpha} \ll u(y^\lambda).$$

Let us discuss the third difficulty. To obtain the symmetry of solutions, one often use either the MMP or the method of moving spheres (MMS). But either

$$(-\Delta)^m u = Q_n^{2m}\Big[\varepsilon\big(\frac{2}{1+|x|^2}\big)^{2m} u + \big(\frac{2}{1+|x|^2}\big)^{\frac{n+2m}{2}+\alpha\frac{n-2m}{2}} u^{-\alpha}\Big]$$

or

$$u(x) = C\int_{\mathbf{R}^n} |x-y|^{2m-n}\Big[\varepsilon\big(\frac{2}{1+|x|^2}\big)^{2m} u + \big(\frac{2}{1+|x|^2}\big)^{\frac{n+2m}{2}+\alpha\frac{n-2m}{2}} u^{-\alpha}\Big] dy$$

contains the weight of $1+|x|^2$, which seems to be difficult to handle using MMS.

[When using the MMS, the center is arbitrary.]

In practice, the MMP can be effectively applied to differential/integral equations with positive exponents. Our case is quite different. Fortunately, we are still successful with the MMP because we have good control on the growth of u, namely $u \in C^\infty(\mathbf{R}^n)$ and

$$\frac{1+|x|^{2m-n}}{C} \leq u(x) \leq C(1+|x|^{2m-n}) \quad \forall x \in \mathbf{R}^n,$$

thanks to

$$u = \big(\frac{2}{1+|x|^2}\big)^{\frac{n-2m}{2}} (v\circ \pi^{-1}) \quad \text{in } \mathbf{R}^n.$$

An application of the Liouville type result is the following Sobolev inequality, which motivates Hang and Yang to work on this higher-order PDE.

A subcritical/critical Sobolev inequality for GJMS's operator on $\mathbb{S}^n$

Let n be an odd number and $m = \frac{n+1}{2}$. Then, for any $\phi \in H^m(\mathbb{S}^n)$ with $\phi > 0$ and any $\alpha \in (0,1) \cup (1, 2n+1)$, we have the following sharp Sobolev inequality

$$\int_{\mathbb{S}^n} \phi \mathbf{P}_n^{2m}(\phi) d\mu_{g_{\mathbb{S}^n}} \geq \frac{\Gamma(n/2+m)}{\Gamma(n/2-m)} |\mathbb{S}^n|^{\frac{\alpha+1}{\alpha-1}} \Big(\int_{\mathbb{S}^n} \phi^{1-\alpha} d\mu_{g_{\mathbb{S}^n}} \Big)^{-\frac{2}{\alpha-1}}. \qquad (12)_\alpha$$

Moreover, equality occurs if ϕ is constant.

[$\alpha = 1$ is the limiting case, the inequality becomes

$$\int_{\mathbb{S}^n} \phi \mathbf{P}_n^{2m}(\phi) d\mu_{g_{\mathbb{S}^n}} \geq \frac{\Gamma(n/2+m)}{\Gamma(n/2-m)} |\mathbb{S}^n| \exp\Big(\frac{2}{|\mathbb{S}^n|} \int_{\mathbb{S}^n} \log \phi d\mu_{g_{\mathbb{S}^n}} \Big), \qquad (13)$$

which can be obtained from $(12)_\alpha$ as $\alpha \searrow 1$.]
[It turns out that

$$(12)_{2n+1} \longrightarrow (12)_\beta \text{ with } \beta \in (1, 2n+1) \longrightarrow (13) \longrightarrow (12)_\alpha \text{ with } \alpha \in (0,1),$$

where the notation $A \longrightarrow B$ means we can obtain B from A.]

The method developed here works equally well for the case of positive exponents, namely we can prove the following.

Theorem (the positive case, namely $n > 2m$)

Let assume $n \geq 3$ be odd and $m < n/2$. Then, under one of the following conditions

1. *either* $\varepsilon \in (0,1)$ *and* $1 < \alpha \leq \dfrac{n+2m}{n-2m}$
2. *or* $\varepsilon = 0$ *and* $1 < \alpha < \dfrac{n+2m}{n-2m}$

any positive, smooth solution v to

$$\mathbf{P}_n^{2m}(v) = Q_n^{2m}(\varepsilon v + v^\alpha) \quad \text{on } \mathbb{S}^n$$

must be constant, hence is equal to $(1-\varepsilon)^{1/(\alpha-1)}$.

[Recall the equation $\mathbf{P}_n^{2m}(v) = Q_n^{2m}(\varepsilon v + v^{-\alpha})$ in the negative case with $m > n/2$, $0 < \alpha \leq (n+2m)/(2m-n)$, and $0 < \varepsilon < \varepsilon_* < 1$.]
[No compactness is required, hence the above result holds for any $0 < \varepsilon < 1$, not necessarily small like $0 < \varepsilon < \varepsilon_* < 1$ in the negative case.]

to negative case

We recall our equation

$$(-\Delta)^m u = Q_n^{2m} \Big[\varepsilon \big(\frac{2}{1+|x|^2} \big)^{2m} u + \big(\frac{2}{1+|x|^2} \big)^{\frac{n+2m}{2} + \alpha \frac{n-2m}{2}} u^{-\alpha} \Big] \quad \text{in } \mathbf{R}^n.$$

In the special case $\varepsilon = 0$ and with $\sigma = \frac{n+2m}{2} + \alpha \frac{n-2m}{2} > 0$ we are led to

$$(-\Delta)^m u = Q_n^{2m} \big(\frac{2}{1+|x|^2} \big)^{\sigma} u^{\frac{n+2m-2\sigma}{n-2m}} \quad \text{in } \mathbf{R}^n. \tag{14}$$

Let us focus on the case $n > 2m$, in particular $Q_n^{2m} > 0$. After normalization, the above equation is similar to the higher-order Hardy-Hénon equation in $\mathbf{R}^n$, namely

$$(-\Delta)^m u = |x|^\sigma u^p \quad \text{in } \mathbf{R}^n.$$

(Sobolev-type critical exponent is $\frac{n+2m+2\sigma}{n-2m}$.) As $\sigma > 0$ and

$$\big(\frac{2}{1+|x|^2} \big)^{\sigma} \sim |x|^{-2\sigma},$$

in this scenario the exponent is 'supercritical' because

$$\frac{n+2m-2\sigma}{n-2m} > \frac{n+2m-4\sigma}{n-2m}.$$

This coincides with the fact that (14) always admits a radial solution. But we should not expect these two types of equations sharing similar properties.

The equation (14), namely

$$(-\Delta)^m u = \Big(\frac{2}{1+|x|^2}\Big)^\sigma u^{\frac{n+2m-2\sigma}{n-2m}} \quad \text{in } \mathbf{R}^n,$$

is also very similar to Matukuma's equation in $\mathbf{R}^3$, namely

$$-\Delta u = \frac{1}{1+|x|^2} u^p \quad \text{in } \mathbf{R}^3$$

with $p > 1$. It is known that this equation admits at least one radial solution for any $p > 1$ (Sobolev's critical exponent is $\frac{2\cdot 3}{3-2} = 6$). If we set $m = 1$, $n = 3$, and $\alpha = -3$, then after normalization our equation (14) becomes

$$-\Delta u = \frac{1}{1+|x|^2} u^3 \quad \text{in } \mathbf{R}^3.$$

So without requiring the exact asymptotic behavior at infinity, it is expected that our equation (14) admits other solutions rather than the radial one. We can also investigate solutions to

$$(-\Delta)^m u = \Big(\frac{2}{1+|x|^2}\Big)^\sigma u^p \quad \text{in } \mathbf{R}^n$$

with

$$n > 2m, \quad p > 1 \quad \text{or} \quad p \geq \frac{n+2m-4\sigma}{n-2m}.$$

For Matukuma's equation in $\mathbf{R}^n$, namely

$$-\Delta u = \frac{1}{1+|x|^2} u^p \quad \text{in } \mathbf{R}^n$$

it is known (after Y. Li, W.M. Ni, E.S. Noussair, C.A. Swanson, E. Yanagida, S. Yotsutani, etc.) that if

$$n = 3, \quad 1 < p < 5 = \frac{3+2}{3-2},$$

then all positive solution must be radially symmetric with respect to the origin. Hence we can ask if such a symmetry result still holds for higher-order cases, at least in the special case

$$(-\Delta)^m u = \Big(\frac{2}{1+|x|^2}\Big)^\sigma u^{\frac{n+2m-2\sigma}{n-2m}} \quad \text{in } \mathbf{R}^n$$

with $n > 2m \geq 4$ and $\sigma > 0$.

Thank you for your attention...

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