# Abstract random polynomial inequalities in Banach spaces

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# Outline

## Random polynomials

- For a multi-index $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{Z}^n$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we let $|\alpha| := |\alpha_1| + \ldots + |\alpha_n|$ and $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. For $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{N}_0^n$, $\alpha! := \alpha_1! \cdots \alpha_n!$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- If $P\colon \mathbb{C}^n \to \mathbb{C}$ is a polynomial given by

$$P(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha, \quad z \in \mathbb{C}^n,$$

  then its degree $\deg(P) := \max\{|\alpha|;\, c_\alpha \neq 0\}$.
- For $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ we denote by $\mathcal{T}_m(\mathbb{T}^n)$ the space of all trigonometric polynomials

$$P(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha, \quad z \in \mathbb{T}^n$$

  on the $n$-dimensional torus $\mathbb{T}^n$ with $\deg(P) := \max\{|\alpha|;\, c_\alpha \neq 0\} \leqslant m$.

The origin of trigonometric random polynomials goes back to the Salem and Zygmund seminal Acta Math. (1954) paper, where they studied trigonometric random polynomials of the type

$$\sum_{n=1}^{\infty} \varepsilon_n c_n \cos(nt + \varphi_n), \quad t \in [-\pi, \pi).$$

This was continued by Kahane (1960), where the study was extended to random polynomials in several variables. In the recent decades the multidimensional variants of the Kahane-Salem-Zygmund inequalities (*KSZ*-inequalities for short) have been of central importance in modern analysis, as, e.g., Fourier analysis, analytic number theory, or holomorphy in high dimensions. The multidimensional *KSZ*-inequality states:

Theorem There exists a positive constant $C$ such that, for each $m$, $n \in \mathbb{N}$ with $m \geqslant 2$ and any trigonometric polynomial $\sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha$ in $\mathcal{T}_m(\mathbb{T}^n)$ there exists a choice of signs $\varepsilon_\alpha = \pm 1$ for which

$$\sup_{z \in \mathbb{T}^n} \Big| \sum_{|\alpha| \leqslant m} \varepsilon_\alpha c_\alpha z^\alpha \Big| \leqslant C \sqrt{n \log m} \Big( \sum_{|\alpha| \leqslant m} |c_\alpha|^2 \Big)^{\frac{1}{2}}.$$

## Extended variant of Kahane-Salem-Zygmund inequality

Theorem For every $a > 0$, each $m, n \geqslant 2$ and all families of $(\varepsilon_\alpha)_{\alpha\in\mathbb{Z}^n, |\alpha|\leqslant m}$ of independent Bernoulli variables on a probability measure space $(\Omega, \mathcal{A}, \mathbb{P})$ and all non-zero $(c_\alpha)_{\alpha\in\mathbb{Z}^n, |\alpha|\leqslant m} \subset \mathbb{C}$, we have

$$\mathbb{P}\Big\{\omega \in \Omega;\ \sup_{z\in\mathbb{T}^n} \Big| \sum_{|\alpha|\leqslant m} \varepsilon_\alpha(\omega) c_\alpha z^\alpha \Big| \geqslant a\sqrt{n \log m}\Big(\sum_{|\alpha|\leqslant m} |c_\alpha|^2\Big)^{\frac{1}{2}}\Big\} \leqslant C(a),$$

where $C(a) := 4\big(\pi^2/(m^{\frac{a^2}{16}} - 1)\big)^n$.

Remark. For all $a > 4\big(\frac{\log(2\pi^2)}{\log 2} + 1\big)^{1/2}$ one has $C(a) < 1$, so for this $a$ we get

$$\mathbb{P}\Big\{\omega \in \Omega;\ \sup_{z\in\mathbb{T}^n} \Big| \sum_{|\alpha|\leqslant m} \varepsilon_\alpha(\omega) c_\alpha z^\alpha \Big| \leqslant a\sqrt{n \log m}\Big(\sum_{|\alpha|\leqslant m} |c_\alpha|^2\Big)^{\frac{1}{2}}\Big\} > 0.$$

In what follows, we shall denote by $\ell_p^n$ the linear space $\mathbb{C}^n$ equipped with the $p$-norm $(1 \leqslant p \leqslant \infty)$.

Theorem (H. P. Boas (2000)) Let $1 \leqslant p \leqslant \infty$ and $m, n \geqslant 2$. Then there exists a choice of signs $(\varepsilon_\alpha)_{|\alpha|=m}$, $\varepsilon_\alpha = \pm 1$ such that

- If $1 \leqslant p \leqslant 2$, then

$$\sup_{z \in B_{\ell_p^{(n)}}} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \Big| \leqslant C \sqrt{mn \log m}\, (m!)^{1-1/p}.$$

- If $2 \leqslant p \leqslant \infty$, then

$$\sup_{z \in B_{\ell_p^n}} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \Big| \leqslant C \sqrt{mn \log m}\; n^{(1/2-1/p)m}\, (m!)^{1/2},$$

where $C > 0$ does not depend on $m$ nor on $n$.

# Abstract random Kahane-Salem-Zygmund inequalities

- We let $(\Omega, \mathcal{A}, \mu)$ to be a measure space and let $X$ be a Banach space. $L^0(\mu, X)$ denotes the space of all equivalence classes of strongly measurable $X$-valued functions on $\Omega$. We let $L^0(\mu) := L^0(\mu, \mathbb{K})$, where $\mathbb{K} := \mathbb{C}$ or $\mathbb{K} := \mathbb{R}$.
- $E \subset L^0(\mu)$ is said to be a Banach function lattice(or space), if there exists $h \in E$ with $h > 0$ a.e. and $E$ is an Banach ideal in $L^0(\mu)$, that is, if $|f| \leqslant |g|$ a.e. with $g \in E$ and $f \in L^0(\mu)$, then $f \in E$ and $\|f\|_E \leqslant \|g\|_E$. By a Banach sequence space we mean a Banach lattice $E \subset \omega(\mathbb{N}) := L^0(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $\mu$ is the counting measure.
- If $E \subset L^0(\mu)$ is a Banach function lattice and $X$ is a Banach space, then the Köthe-Bochner space $E(X)$ consists of of all $f \in L^0(\mu, X)$ with $\|f(\cdot)\|_X \in E$, and is equipped with the norm $\|f\|_{E(X)} := \| \|f(\cdot)\|_X\|_E$.

- (A. Defant–M. M.) Given a Banach function space $X$ over a probability measure space $(\Omega, \mathcal{A}, \mathbb{P})$ and a sequence of random variables $(\gamma_i)_{i\in\mathbb{N}} \subset X$, we are looking for a function $\psi\colon \mathbb{N} \to (0,\infty)$ and a sequence $(S^n)$,

$$S^n := (\mathbb{K}^n, \|\cdot\|_n),\ n\ \in\ \mathbb{N}$$

of semi-normed spaces such that, for each $N, K$ and for every choice of finitely many vectors $a_i := (a_i(j))_{j=1}^N \in \ell_\infty^N$, $1 \leqslant i \leqslant K$, we have

$$\Big\|\sum_{i=1}^K a_i\gamma_i\Big\|_{X(\ell_\infty^N)} \leqslant \psi(N) \sup_{1\leqslant j\leqslant N} \big\|(a_i(j))_{i=1}^K\big\|_{S^K},$$

that is,

$$\Big\|\sup_{1\leqslant j\leqslant N}\Big|\sum_{i=1}^K a_i(j)\gamma_i\Big|\Big\|_X \leqslant \psi(N) \sup_{1\leqslant j\leqslant N} \big\|(a_i(j))_{i=1}^K\big\|_{S^K}.$$

- A sequence $(\gamma_i)_{i\in\mathbb{N}} \subset X$ is said to satisfy the $KSZ$-inequality of type $(X,(S^n),\psi)$ provided that the above inequality holds.

## *KSZ*-inequalities by lattice constants

- If $X$ is a Banach lattice, then for each $n \in \mathbb{N}$, the $M$-constant $\mu_n(X)$ is defined by

$$\mu_n(X) := \sup\Big\{\big\| \sup_{1\leqslant j\leqslant n} |x_j|\big\|_X : \; \|x_j\|_X \leqslant 1, \text{ for } 1 \leqslant j \leqslant n\Big\}.$$

- Properties: $(\mu_n(X))_n$ is a non-decreasing sequence with $\mu_n(X) \in [1, n]$ for each $n \in \mathbb{N}$; $(\mu_n(X))_n$ is a submultiplicative sequence, that is,

$$\mu_{mn}(X) \leqslant \mu_m(X)\mu_n(X), \quad m, n \in \mathbb{N};$$

- $\left(\frac{\mu_n(X)}{n}\right)$ is non-increasing sequence (Abramovich–Lozanovskii (1973)).
- $\lim_{n\to\infty} \frac{\mu_n(X)}{n} \in \{0, 1\}$. This implies $\mu_n(X) = n$ for each $n \in \mathbb{N}$ whenever $\lim_{n\to\infty} \frac{\mu_n(X)}{n} = 1$.
- Theorem (Abramovich–Lozanovskii (1973)) If $\lim_{n\to\infty} \frac{\mu_n(X)}{n} = 0$, then all odd duals of $X$ are $KB$-spaces (Kantorovich-Banach spaces).

Proposition (A. Defant–M. M) Let $X$ be a Banach lattice over $(\Omega, \mathcal{A}, \nu)$ and let $\psi\colon \mathbb{N} \to [1, \infty)$ be given by $\psi(n) := \mu_n(X)$ for each $n \in \mathbb{N}$. Then every sequence $(\gamma_i)_{i\in\mathbb{N}}$ of random variables in $X$ satisfies the *KSZ*-inequality of type $(X, (S^n), \psi)$,

$$\Big\| \sup_{1\leqslant j\leqslant N} \Big| \sum_{i=1}^{K} a_i(j)\gamma_i \Big| \Big\|_X \leqslant \psi(N) \sup_{1\leqslant j\leqslant N} \big\|(a_i(j))_{i=1}^K\big\|_{S^K}, \quad (a_i(j))_{j=1}^N \in \ell_\infty^N$$

with $S^n := (\mathbb{K}^n, \|\cdot\|_n)$, where the semi-norm $\|\cdot\|_n$ (resp., norm $\|\cdot\|_n$, whenever the $\gamma_i$ are linearly independent) are defined by

$$\|z\|_n := \|z_1\gamma_1 + \ldots + z_n\gamma_n\|_X, \quad z = (z_1, \ldots, z_n) \in \mathbb{K}^n\,.$$

## $M$-constants for some class of Orlicz spaces

Let $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an Orlicz function (that is, a convex, increasing and continuous positive function with $\Phi(0) = 0$). The Orlicz space $L_\Phi$ over a measure space $(\Omega, \mathcal{A}, \mu)$ is defined to be the space of all $f \in L^0(\mu)$ such that $\int_\Omega \Phi(\lambda|f|)\, d\mu < \infty$ for some $\lambda > 0$, and it is equipped with the norm

$$\|f\|_\Phi := \inf\left\{\lambda > 0;\ \int_\Omega \Phi\Big(\frac{|f|}{\lambda}\Big)\, d\mu \leqslant 1\right\}.$$

For $1 \leqslant r < \infty$, the exponential Orlicz function $\varphi_r(t) := e^{t^r} - 1$, $t \geqslant 0$.

Theorem. [A. Defant– M. M] Let $L_\Phi$ be an Orlicz space over a probability measure space $(\Omega, \mathcal{A}, \nu)$ with $\Phi(t) := e^{\varphi(t)} - 1$ for all $t \geqslant 0$, where $\varphi$ is an Orlicz function on $\mathbb{R}_+$ with, for some $\gamma > 0$, $\varphi(st) \leqslant \gamma\varphi(s)\varphi(t)$ for all $s \in (0, 1]$ and $t > 0$. Then, for each $n \in \mathbb{N}$, one has

$$\mu_n(L_\Phi) \leqslant \frac{C}{\varphi^{-1}\big(\varphi(1)/(1 + \log n)\big)},$$

where $C = (e - 1)\gamma\varphi(1)$.

Corollary For $r \in [1, \infty)$ let $L_{\varphi_r}$ be an Orlicz space over a probability measure space $(\Omega, \mathcal{A}, \nu)$ with $\varphi_r(t) = e^{t^r} - 1$ for all $t \geqslant 0$. Then for each $n \in \mathbb{N}$ one has

$$\mu_n(L_{\varphi_r}) \leqslant (e-1)(1+\log n)^{\frac{1}{r}}\,.$$

## *KSZ*-inequalities for subgaussian random variables

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $f$ a random variable. If $f$ is real-valued, then $f$ is said to be subgaussian, whenever there exists $s \geqslant 0$ such that

$$\mathbb{E}\exp(\lambda f) \leqslant \exp\left(\frac{s^2\lambda^2}{2}\right), \quad \lambda \in \mathbb{R},$$

and if $f$ is complex-valued, whenever there exists $s \geqslant 0$ such that

$$\mathbb{E}\exp(\mathrm{Re}(zf) \leqslant \exp\left(\frac{s^2|z|^2}{2}\right), \quad z \in \mathbb{C}.$$

The best such $s$ is denoted by $\mathrm{sg}(f)$.

- A real-valued sequence $(f_n)$ is called subgaussian if there is $s \geqslant 0$ such that for any $x = (x_n) \in \ell_2$ of norm one, the random variable $f = \sum_{n=1}^{\infty} x_n f_n$ is subgaussian. The best possible number $s$ is denoted by $sg((f_n))$.

- A complex-valued sequence $(f_n)$ is said to be subgaussian, whenever $(\mathrm{Re} f_n)$ and the imaginary parts $(\mathrm{Im} f_n)$, is subgaussian.

## Examples

- Every sequence $(\gamma_n)$ of independent, real (resp., complex) normal gaussian variables is subgaussian with $sg((\gamma_n)) = 1$.
- Every sequences $(\varepsilon_n)$ of independent Rademacher variables is subgaussian with $\mathrm{sg}((\varepsilon_n)) = 1$.

Theorem (A. Defant–M. M.) Let $(\gamma_i)_{i\in\mathbb{N}}$ be a (real or complex) subgaussian sequence of random variables over $(\Omega, \mathcal{A}, \mathbb{P})$ with $s = \mathrm{sg}((\gamma_i))$. The following statements are true for each $K, N \in \mathbb{N}$ and all $a_1, \ldots, a_K \in \ell_\infty^N$ with $a_i = (a_i(j))_{j=1}^N$, $1 \leqslant i \leqslant K$:

(1) There is a constant $C_2 = C(s) > 0$ such that

$$\Big\| \sum_{i=1}^K \gamma_i a_i \Big\|_{L_{\varphi_2}(\ell_\infty^N)} \leqslant C_2 (1+\log N)^{\frac{1}{2}} \sup_{1\leqslant j\leqslant N} \big\| \big(a_i(j)\big)_{i=1}^K \big\|_{\ell_2^K}.$$

(2) If in addition $M = \sup_i \|\gamma_i\|_\infty < \infty$, then for every $r \in (2,\infty)$ there is a constant $C_r = C(r,s,M) > 0$ such that for $1/r' := 1 - 1/r$, we have

$$\Big\| \sum_{i=1}^K \gamma_i a_i \Big\|_{L_{\varphi_r}(\ell_\infty^N)} \leqslant C_r (1+\log N)^{\frac{1}{r}} \sup_{1\leqslant j\leqslant N} \big\| \big(a_i(j)\big)_{i=1}^K \big\|_{\ell_{r',\infty}^K}.$$

- Here $\ell_{p,\infty}$ for $p \in (1,\infty)$ denotes the Marcinkiewicz sequence space of all scalar sequences $x = (x_k)_k \in \omega(\mathbb{N})$ equipped with the norm

$$\|x\|_{p,\infty} := \sup_{n\in\mathbb{N}} \frac{x_1^* + \ldots + x_n^*}{n^{1-\frac{1}{p}}},$$

where $(x_k^*)$ denotes the decreasing rearrangement of the sequence $(|x_k|)$.

## Variants of Kahane–Salem–Zygmund inequality

Let $P$ be an $m$-homogeneous random Bernoulii polynomial over a probability measure $(\Omega, \mathcal{A}, \mathbb{P})$ given by

$$P(\omega, z) := \sum_{|\alpha|=m} \varepsilon_\alpha(\omega) c_\alpha z^\alpha, \quad \omega \in \Omega,\ z \in \mathbb{C}^n.$$

Theorem (F. Bayart (2012)) For an arbitrary $n$-dimensional Banach space $X_n = (\mathbb{C}^n, \|\cdot\|)$ and for every $r \in [2, \infty)$ one has

$$\mathbb{E}\Big(\sup_{z\in B_{X_n}} \big|P(\cdot, z)\big|\Big) \leqslant C_r\big(n(1+\log m)\big)^{\frac{1}{r}} \sup_{|\alpha|=m} |c_\alpha| \Big(\frac{\alpha!}{m!}\Big)^{\frac{1}{r'}} \sup_{z\in B_{X_n}} \Big(\sum_{k=1}^n |z_k|^{r'}\Big)^{\frac{m}{r'}},$$

where $C_r > 0$ is a constant only depending on $r$.

Given a real number $1 \leqslant \lambda < \infty$. A Banach space $X$ $\lambda$-embeds into a Banach $Y$ whenever there exists an isomorphic embedding $T$ of $X$ into $Y$ such

$$\|T\|_{X \to Y} \, \|T^{-1}\|_{T(X) \to X} \leqslant \lambda \,.$$

In this case, we call $T$ a $\lambda$-embedding of $X$ into $Y$.

Theorem [A. Defant–M. M.] For every $r \in [2, \infty)$ there is a constant $C_r > 0$ such that, for every finite-dimensional Banach space $E$, for every $\lambda$-embedding $I \colon E \to \ell_\infty^N$, and for every choice of $x_1, \ldots, x_K \in E$, we have

$$\Big\| \sum_{i=1}^{K} \gamma_i x_i \Big\|_{L_{\varphi_r}(E)} \leqslant C_r \, \|I^{-1}\| \, (1 + \log N)^{\frac{1}{r}} \sup_{1 \leqslant j \leqslant N} \big\| (I(x_i)(j))_{i=1}^{K} \big\|_{S_{r'}^K} \,,$$

Theorem (A. Defant–M. M.) For every $2 \leqslant r < \infty$, there exists a constant $C_r > 0$ such that, for any choice of polynomials $P_1, \ldots, P_K \in \mathcal{T}_m(\mathbb{C}^n)$, we have

$$\Big\| \sup_{z \in \mathbb{T}^n} \Big| \sum_{i=1}^{K} \varepsilon_i P_i(z) \Big| \Big\|_{L_{\varphi_2}} \leqslant C_2 \big( n(1 + \log m) \big)^{\frac{1}{2}} \sup_{z \in \mathbb{T}^n} \big\| (P_i(z))_{i=1}^{K} \big\|_{\ell_2},$$

and for $2 < r < \infty$

$$\Big\| \sup_{z \in \mathbb{T}^n} \Big| \sum_{i=1}^{K} \varepsilon_i P_i(z) \Big| \Big\|_{L_{\varphi_r}} \leqslant C_r \big( n(1 + \log m) \big)^{\frac{1}{r}} \sup_{z \in \mathbb{T}^n} \big\| (P_i(z))_{i=1}^{K} \big\|_{\ell_{r',\infty}}.$$

Theorem (Defant–M. M.) For every $r \in [2, \infty)$ there is a constant $C_r > 0$ such that for each $m \in \mathbb{N}_0, n \in \mathbb{N}$, every complex $n$-dimensional Banach space $X$, and every choice of polynomials $P_1, \ldots, P_K \in \mathcal{P}_m(X)$, we have

$$\Big\| \sup_{z \in B_X} \Big| \sum_{i=1}^{K} \gamma_i P_i(z) \Big| \Big\|_{L_{\varphi_r}} \leqslant C_r \big( n(1 + \log m) \big)^{\frac{1}{r}} \sup_{z \in B_X} \big\| (P_i(z))_{i=1}^{K} \big\|_{S_{r'}^K},$$

where $S_{r'}^K := \ell_2^K$ for $r = 2$ and $S_{r'}^K := \ell_{r',\infty}^K$ for $r \in (2, \infty)$.

The proof is based on the following result.

Proposition (A. Defant–M. M.) Let $X$ be an $n$-dimensional Banach space, and $K \subset B_X$ a convex and compact subset, which satisfies a Markov–Fréchet inequality with exponent $\nu$ and constant $M$. For each $m \in \mathbb{N}$ there exists a subset $F \subset K$ such that

$$\|P\|_K \leqslant 2 \sup_{z \in F} |P(z)\|_F, \quad P \in \mathcal{P}_m(X),$$

with card $F \leqslant N$, where $N = \big(1 + 2Mm^\nu\big)^n$ if $X$ is real and $N = \big(1 + 2Mm^\nu\big)^{2n}$ if $X$ is complex space. In other words the Banach space $\mathcal{P}_m(X)$, 2-embeds into $\ell_\infty^N$.

Given a Banach space $X$ and a nonempty compact subset $K \subset B_X$.

Definition. We say that $K$ satisfies a Markov–Fréchet inequality whenever there is an exponent $\nu \geqslant 0$, and a constant $M > 0$ such that for all $P \in \mathcal{P}(X)$ one has

$$\sup_{z \in K} \|\nabla P(z)\|_{X^*} \leqslant M(\deg P)^{\nu} \sup_{z \in K} |P(z)|,$$

where $\nabla P(z) \in X^*$ denotes the Fréchet derivative of $P$ in $z \in K$. If this inequality only holds for a subclass $\mathcal{P}$ of $\mathcal{P}(X)$, then we say that $K$ satisfies a Markov-Fréchet inequality for $\mathcal{P}$ with exponent $\nu$ and constant $M$.

Theorem (Harris (1997)) Let $X$ be a complex Banach space. Then $B_X$ satisfies a Markov–Fréchet inequality with constant $e$ and exponent $\nu = 1$.

## Random Dirichlet polynomials

Combining Bohr's vision of ordinary Dirichlet series and the mentioned results, we provide some new *KSZ*–inequalities for random Dirichlet polynomials. Some inequalities of this type recently played a crucial role within the study of Dirichlet series.

Given a finite subset $A \subset \mathbb{N}$, we denote by $\mathcal{D}_A$ the Banach space of all Dirichlet polynomials $D$ given by

$$D(s) := \sum_{n \in A} a_n n^{-s}, \quad s \in \mathbb{C}\,,$$

with $\{a_n\}_{n \in A} \subset \mathbb{C}$. Since each such Dirichlet polynomial defines a bounded and holomorphic function on the right half-plane in $\mathbb{C}$, the space $\mathcal{D}_A$ forms a Banach space equipped with the norm

$$\|D\|_\infty := \sup_{\mathrm{Re} s>0} \Big| \sum_{n=1}^{N} a_n n^{-s} \Big| = \sup_{t \in \mathbb{R}} \Big| \sum_{n=1}^{N} a_n n^{-it} \Big|.$$

Remark We note that the particular cases $a_n = 1$ and $a_n = (-1)^n$ play a crucial role within the study of the Riemann zeta-function $\zeta\colon \mathbb{C}\setminus\{1\} \to \mathbb{C}$. In fact, in recent times, techniques related to random inequalities for Dirichlet polynomials have gained more and more importance. This may be illustrated by a deep classical result of Turán (1962), which states that the truth of the famous Lindelöf's conjecture:

$$\zeta\big(1/2 + it\big) = \mathcal{O}_\varepsilon(t^\varepsilon), \quad t \in \mathbb{R}\,,$$

with an arbitrarily small $\varepsilon > 0$, is equivalent to the validity of the inequality:

$$\Big|\sum_{n=1}^{N} \frac{(-1)^n}{n^{it}}\Big| \leqslant C\, N^{\frac{1}{2}+\varepsilon}(2 + |t|)^\varepsilon, \quad t \in \mathbb{R}$$

for an arbitrarily small $\varepsilon > 0$ and with $C = C(\varepsilon)$.

In order to formulate our main result we need two characteristics of the finite set $A \subset \mathbb{N}$ defining $\mathcal{D}_A$. As usual, for $x \geqslant 2$, we denote by $\pi(x)$ the number of all primes in the interval $[2, x]$, and by $\Omega(n)$ the number of prime divisors of $n \in \mathbb{N}$ counted accorded to their multiplicities. We define

$$\Pi(A) := \max_{n \in A} \pi(n), \qquad \Omega(A) := \max_{n \in A} \Omega(n) .$$

Theorem (A. Defant–M. M.) For every $r \in [2, \infty)$ there is a constant $C_r > 0$ such that for any finite set $A \subset \mathbb{N}$ and any choice of Dirichlet polynomials $D_1, \ldots, D_K \in \mathcal{D}_A$, we have

$$\Big\| \sup_{t \in \mathbb{R}} \Big| \sum_{j=1}^{K} \gamma_j D_j(t) \Big| \Big\|_{L_{\varphi_r}} \leqslant C_r \Big( 1 + \Pi(A) \big( 1 + 20 \log \Omega(A) \big) \Big)^{\frac{1}{r}} \sup_{t \in \mathbb{R}} \big\| (D_j(t))_{j=1}^{K} \big\|_{S_{r'}} .$$

Corollary. For every $r \in [2, \infty)$ there is a constant $C_r > 0$ such that such, for every Dirichlet random polynomial $\sum_{n \in A} \gamma_n a_n n^{-it}$ in $\mathcal{D}_A$ one has

$$\Big\| \sup_{t \in \mathbb{R}} \Big| \sum_{n \in A} \gamma_n a_n n^{-it} \Big| \Big\|_{L_{\varphi_r}} \leqslant C_r \Big( 1 + \Pi(A) \big( 1 + 20 \log \Omega(A) \big) \Big)^{\frac{1}{r}} \big\| (a_n)_{n \in A} \big\|_{S_{r'}} .$$

Idea of proof:

- We embed $\mathcal{D}_A$ into a certain space of trigonometric polynomials, controlling the degree as well as the number of variables of the polynomials in this space. To achieve this, we use the so-called Bohr lift:

$$\mathcal{B}_A \colon \mathcal{D}_A \to \mathcal{T}_{\Omega(A)}(\mathbb{T}^{\Pi(A)}), \ \sum_{n \in A} a_n n^{-s} \mapsto \sum_{\alpha : \mathfrak{p}^\alpha \in A} a_{\mathfrak{p}^\alpha} z^\alpha .$$

  By Kronecker's theorem on Diophantine approximation we know that the continuous homomorphism

$$\beta \colon \mathbb{R} \to \mathbb{T}^{\Pi(A)}, \ t \to \big( \mathfrak{p}_k^{it} \big)_{k=1}^{\Pi(A)}$$

  has dense range. This implies that $\mathcal{B}_A$ is an isometry into.

- There is a subset $F \subset \mathbb{T}^{\Pi(A)}$ with $\mathrm{card}(F) \leqslant N = (1 + 20\,\Omega(A))^{\Pi(A)}$ such that

$$I\colon \mathcal{T}_{\Omega(A)}(\mathbb{T}^{\Pi(A)}) \hookrightarrow \ell_\infty^N, \;\; I(P) := (P(z_i))_{i \in F},$$

  is a 2-isomorphic embedding. Combining all these facts we get the above theorem.

In the following example we consider interesting subclass of Dirichlet polynomials of length $N$, each given by a particular finite subset $A \subset \mathbb{N}$:

Example. For $N \in \mathbb{N}$ and $2 \leqslant x \leqslant N$ define

$$A(N, x) := \{1 \leqslant n \leqslant N;\, \pi(n) \leqslant x\}\,.$$

Then $\mathcal{D}_{A(N,x)}$ is the space of all Dirichlet polynomials of length $N$, which only 'depend on $\pi(x)$-many primes'. Using remarkable estimates for $\pi(x)$ due to Costa Periera (1985):

$$\frac{x \log 2}{\log x} < \pi(x), \quad x \geqslant 5 \quad \text{and} \quad \pi(x) < \frac{5x}{3 \log x}, \qquad x > 1\,,$$

we get $\Pi(A(N,x)) \leqslant \pi(x) < \frac{5x}{3 \log x}$. Since for each $1 \leqslant n = p^{\alpha} \leqslant N$ with $\alpha \in \mathbb{N}^{\pi(x)}$ we have $2^{|\alpha|} \leqslant N$, it follows that

$$\Omega(A(N,x)) \leqslant \frac{\log N}{\log 2}\,.$$

With these estimates for $\Pi(A(N,x))$ and $\Omega(A(N,x))$ our *KSZ*–inequalities extend Queffélec's results (1995).

In the special case $x = N$, we denote by $\mathcal{D}_N$ the Banach space of all Dirichlet polynomials of length $N$, in other words, $\mathcal{D}_N = \mathcal{D}_{A(N)}$ with $A(N) = \{1, \ldots, N\}$. Then

$$\Pi(A(N)) < \frac{5N}{3\log N}, \quad \Omega(A(N)) \leqslant \frac{\log N}{\log 2}.$$

It is worth noting that in the case $N = p_n$, the $n$th prime, one has $\Pi(A(N)) = n$.

## Random multilinear forms in Banach spaces

- Given Banach spaces $X_1, \ldots, X_m$, the Banach space $\mathcal{L}_m(X_1, \ldots, X_m)$ of all scalar-valued $m$-linear bounded mappings $L$ on $X_1 \times \cdots \times X_m$ is equipped with the norm

$$\|L\| := \sup\left\{|L(x_1, \ldots, x_m)| : x_j \in B_{X_j}, 1 \leqslant j \leqslant m\right\}.$$

- For a given Banach space $X$ and $m \in \mathbb{N}$, we denote by $\mathcal{P}_m(X)$ the Banach space of all polynomials $P$ on $X$ of degree $m$ (i.e., there is $L \in \mathcal{L}_m(X, \ldots, X)$ such that $P(x) = L(x, \ldots, x)$ for all $x \in X$) equipped with the norm

$$\|P\| := \sup\{|P(z)| : z \in B_X\}.$$

We let $\|P\|_E := \sup\{|P(z)|; z \in E\}$, whenever $E$ is a non-empty subset of $X$.

Applying our techniques to spaces of multilinear forms on finite dimensional Banach spaces, we can state the following theorem.

Theorem (A. Defant–M. M.) For every $r \in [2, \infty)$ there is a constant $C_r > 0$ such that, for every choice of finite dimensional Banach spaces $X_j$ with $\dim X_j = n_j$, $1 \leqslant j \leqslant m$, and $m$-linear mappings $L_1, \ldots, L_K \in \mathcal{L}_m(X_1, \ldots, X_m)$, one has

$$\Big\| \sup_{(z_1,\ldots,z_m)\in B_{X_1\times\cdots\times X_m}} \Big| \sum_{i=1}^{K} \gamma_i L_i(z_1, \ldots, z_m)\Big| \Big\|_{L_{\varphi_r}}$$

$$\leqslant C_r \Big( \sum_{j=1}^{m} n_j (1 + \log m)\Big)^{\frac{1}{r}} \sup_{(z_1,\ldots,z_m)\in B_{X_1\times\cdots\times X_m}} \big\| (L_i(z_1, \ldots, z_m))_{i=1}^{K} \big\|_{S_{r'}^K},$$

where $S_{r'}^K := \ell_2^K$ for $r = 2$ and $S_{r'}^K := \ell_{r',\infty}^K$ for $r \in (2, \infty)$.

The proof of the above theorem is based on the following result.

Proposition (A. Defant – M. M.) Let $X_j$ with $\dim X_j = n_j, 1 \leqslant j \leqslant m$ be finite dimensional (real or complex) Banach spaces. Then there is a subset $F \subset \prod_{j=1}^m B_{X_j}$ of cardinality

$$\mathrm{card}(F) \leqslant \prod_{j=1}^{m} \left(1 + 2m\right)^{2n_j}$$

such that for every $L \in \mathcal{L}_m(X_1, \dots, X_m)$,

$$\|L\|_\infty \leqslant 2 \sup_{(z_1,\dots,z_m)\in F} |L(z_1, \dots, z_m)| \,.$$

If all Banach spaces $X_j$ are real, we may replace the exponents $2n_j$ by $n_j$.

## Polynomial inequalities via random processes

- Given a pseudo-metric $(T, d)$, we denote by $N(T, d; \varepsilon)$ the entropy function associated with the pseudo-metric $d$ on the set $T$ for $\varepsilon > 0$, i.e.,

$$N(T, d; \varepsilon)$$

is the smallest number of open balls of radius $\varepsilon > 0$ in the pseudo-metric $d$ needed to cover the set $T$.

- Let $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an Orlicz function. The entropy integral of $(T, d)$ with respect to $\Phi$ is defined by

$$J_\Phi(T, d) = \int_0^{\Delta(T)} \Phi^{-1}\big(N(T, d; \varepsilon)\big)\, d\varepsilon,$$

where $\Delta(T) = \sup_{s,t\in T} d(s, t)$ denotes the diameter of $T$.

- If $(X_t)_{t\in T}$ is a stochastic process where $T$ is an index set. Then

$$\mathbb{E}\big(\sup_{t\in T} X_t\big) := \sup\Big\{\mathbb{E}\big(\sup_{t\in F} X_t\big):\ F\subset T,\ F \text{ finite}\Big\},$$

where the right-hand side makes sense as soon as r.v. $X_t$ is integrable for every $t\in T$.

- A fundamental example of stochastic processes is a random series

$$X_t = \sum_{k\geqslant 1} \xi_k f_k(t),$$

where $f_k$ are functions defined on a set $T$ and $\xi_k$ are independent random variables on a measure space $(\Omega, \mathcal{A}, \mu)$.

- The basis example is the random Fourier series,

$$X_t = \sum_{k\geqslant 1} \xi_k e^{2\pi ikt}, \quad t\in[0,1].$$

- Pisier's Theorem If $(X_t)_{t\in T}$ is a stochastic process in the Orlicz space $L_\Phi(\Omega, \mathcal{A}, \mathbb{P})$ on a probability measure space such that

$$\|X_s - X_t\|_\Phi \leqslant d(s,t), \quad s,t \in T,$$

then we have

$$\mathbb{E}\big(\sup_{s,t\in T} |X_s - X_t|\big) \leqslant CJ_\Phi(T,d)$$

for some absolute constant $C > 0$.

- A. Defant, D. Galicer, M. Mansilla, M. M., S. Muro, **Projection constants for spaces of multivariate polynomials**, 2022, 181 pp. (Preprint).