The Evolution to Equilibrium of Nonlinear Fokker–Planck–Kolmogorov Equations

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Work supported by Deutsche Forschungsgemeinschaft (DFG) through "Collaborative Research Centre (CRC) 1283".

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Contents

1. From nonlinear FPKE to distribution dependent SDE (= McKean–Vlasov SDE): general scheme

Let $\mathcal{P}(\mathbb{R}^d)$ denote the set of all Borel probability measures on $\mathbb{R}^d$ and let

$$b = (b_1, \dots, b_d) \colon [0, \infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \longrightarrow \mathbb{R}^d,$$
$$\sigma = (\sigma_{ij})_{1 \le i,j \le d} \colon [0, \infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \longrightarrow L(\mathbb{R}^d, \mathbb{R}^d)$$

be measurable. Consider

$$\mathrm{d}X(t) = b\big(t, X(t), \mathcal{L}_{X(t)}\big)\,\mathrm{d}t + \sigma\big(t, X(t), \mathcal{L}_{X(t)}\big)\,\mathrm{d}W(t), \tag{DDSDE}$$

where $W(t)$, $t \geq 0$, is an $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and

$$\mathcal{L}_{X(t)} := \mathbb{P} \circ X(t)^{-1}, \quad t \geq 0,$$

are the time marginal laws of $X(t)$, $t \geq 0$, under $\mathbb{P}$.

1. From nonlinear FPKE to distribution dependent SDE (= McKean–Vlasov SDE): general scheme

Remark

DDSDEs (also called McKean–Vlasov SDEs) have a very long history and probabilistically weak and strong solutions as well as solutions to the corresponding martingale problems have been constructed and their uniqueness has been shown under various conditions.

See e.g. the classical papers McKean 1966, 1967, Sznitman 1984, Funaki 1984, Scheutzow 1987 among many others and more recent work on arXiv, as e.g. Chassagneux/Crisan/Delarue 2017, Mishura/Veretennikov 2017, F.Y. Wang 2017, Huang/R./F.Y. Wang 2017, Hammersley/Siska/Szpruch 2018, dos Reis/Smith/Tankov 2018, X. Zhang/R. 2018, Mehri/Stannat 2018 and many *others ...*

1. From nonlinear FPKE to distribution dependent SDE (= McKean–Vlasov SDE): general scheme

By Itô's formula it is easy to find the nonlinear (!) Fokker–Planck–Kolmogorov equation (FPKE for short) for the **time marginal laws** $\boldsymbol{\mathcal{L}_{X(t)} =: \mu_t}$, $\boldsymbol{t} \geq 0$, of the solution $X(t)$, $t \geq 0$, to (DDSDE). More precisely, for smooth $\varphi : \mathbb{R}^d \to \mathbb{R}$ with compact support we have for $t \geq 0$

$$
\begin{aligned}
\int_{\mathbb{R}^d} \varphi(x)\mu_t(\mathrm{d}x) &= \int_\Omega \varphi(X(t)(\omega))\,\mathbb{P}(\mathrm{d}\omega) \\
&= \int_\Omega \varphi(X(0)(\omega))\,\mathbb{P}(\mathrm{d}\omega) + \int_\Omega \int_0^t L_{\mathcal{L}_{X(s)}}\varphi(X(s)(\omega))\,\mathrm{d}s\,\mathbb{P}(\mathrm{d}\omega) \\
&= \int_{\mathbb{R}^d} \varphi(x)\mu_0(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^d} L_{\mu_s}\varphi(s,x)\mu_s(\mathrm{d}x)\mathrm{d}s \qquad \text{(NLFPKE)}
\end{aligned}
$$

where for $x \in \mathbb{R}^d$, $t \geq 0$, and $a_{ij} := (\sigma\sigma^T)_{ij}$, $1 \leq i,j \leq d$,

$$
L_{\mu_t}\varphi(t,x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,x,\mu_t)\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x) + \sum_{i=1}^d b_i(t,x,\mu_t)\frac{\partial}{\partial x_i}\varphi(x).
$$

1. From nonlinear FPKE to distribution dependent SDE (= McKean–Vlasov SDE): general scheme

We refer to Chap. 10 in: Bogachev/Krylov/R./Shaposhnikov: Fokker–Planck–Kolmogorov Equations, AMS Monograph 2015, pp. 491.

We can rewrite (NLFPKE) in the sense of Schwartz distributions as follows:

$$\frac{\partial}{\partial t}\mu_t = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}\left[a_{ij}(t,x,\mu_t)\mu_t\right] - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left[b_i(t,x,\mu_t)\mu_t\right],$$

$$\mu_0 \in \mathcal{P}(\mathbb{R}^d) \text{ given},$$

or shortly

$$\partial_t \mu = \frac{1}{2}\partial_i \partial_j (a_{ij}(\mu)\mu) - \partial_i (b_i(\mu)\mu),$$

$$\mu_0 \in \mathcal{P}(\mathbb{R}^d) \text{ given}.$$

1. From nonlinear FPKE to distribution dependent SDE ($=$ McKean–Vlasov SDE): general scheme

Now let us go backwards, i.e. first solve (NLFPKE) **and then construct a weak solution to** (DDSDE)**.**

Let a_{ij}, b_i, $1 \leq i, j \leq d$, be as in the previous section.

Assumption: There **exists** a solution $[0, \infty) \ni t \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d)$ of (NLFPKE) such that

(i) For all $T > 0$ and $1 \leq i, j \leq d$

- $a_{ij}, b_i \in L^1([0, T] \times U, \mu_t \mathrm{d}t)$ for every ball $U \subset \mathbb{R}^d$,
- $\displaystyle\int_0^T \int_{\mathbb{R}^d} \frac{|a_{ij}(t, x, \mu_t)| + |\langle x, b_i(t, x, \mu_t)\rangle|}{1 + |x|^2} \, \mu_t(\mathrm{d}x) \, \mathrm{d}t < \infty$

(ii) $[0, \infty) \ni t \mapsto \mu_t$ is weakly continuous.

1. From nonlinear FPKE to distribution dependent SDE (= McKean–Vlasov SDE): general scheme

Now **fix** this solution $(\mu_t)_{t\geq 0}$.

Theorem I ([Barbu/R. 2018, SIAM Journal of Math. Analysis 2018 and Ann. Probab. 2020])

There exists a d-dimensional $(\mathcal{F}_t)$*-Brownian motion* $W(t)$, $t \geq 0$, *on a stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *and a continuous* $(\mathcal{F}_t)$*-progressively measurable map* $X : [0, \infty) \times \Omega \to \mathbb{R}^d$ *satisfying the following (DD)SDE*

$$\mathrm{d}X(t) = b\big(t, X(t), \mu_t\big)\,\mathrm{d}t + \sigma\big(t, X(t), \mu_t\big)\,\mathrm{d}W(t),$$

where $\sigma = ((a_{ij})_{1\leq i,j\leq d})^{\frac{1}{2}}$, *such that we have, for the marginals,*

$$\mathcal{L}_{X(t)} = \mathbb{P}\circ X(t)^{-1} = \mu_t, \quad t \geq 0. \tag{PR}$$

1. From nonlinear FPKE to distribution dependent SDE (= McKean–Vlasov SDE): general scheme

Proof. Let $(\mu_t)_{t\geq 0}$ be as in **Assumption**. Then by [Bogachev/R./Shaposhnikov 2019], which is a recent regeneralization of a beautiful result in [Trevisan: EJP 2016], there exists a probability measure P on $C([0,T];\mathbb{R}^d)$ equipped with its Borel σ-algebra and its natural filtration generated by the evaluation maps π_t, $t \in [0,T]$, defined by

$$\pi_t(w) := w(t),\ \ w \in C([0,T],\mathbb{R}^d),$$

solving the martingale problem for the **linear** Kolmogorov operator (with $\mu = (\mu_t)_{t\geq 0}$ as above fixed)

$$L_u := \tfrac{1}{2} a_{ij}(\mu)\partial_i\partial_j + b_i(\mu)\partial_i$$

with marginals

$$P \circ \pi_t^{-1} = \mu_t, \quad t \geq 0.$$

Then, the assertion follows by a standard result (see e.g. [Stroock: LMS Text 1987]).

2. The Nemytskii – case

The dependence of a_{ij} and b_i, $1 \leq i, j \leq d$, on the measure $\mu_t(\mathrm{d}x)$ can be arbitrary (as long as it is measurable). In Section 3 we shall, however, consider examples of the following type: we look for a solution $(\mu_t)_{t\geq 0}$ to (NLFPKE), which is absolutely continuous, i.e.

$$\mu_t(\mathrm{d}x) = u(t,x)\,\mathrm{d}x, \quad t \geq 0,$$

($\mathrm{d}x$ = Lebesgue measure on $\mathbb{R}^d$) and a_{ij}, b_i are of Nemytskii–type, i.e. for $t \geq 0$, $x \in \mathbb{R}^d$,

$$\begin{aligned} a_{ij}(t,x,u(t,\cdot)\,\mathrm{d}x) &= \overline{a_{ij}}(t,x,u(t,x)), \\ b_i(t,x,u(t,\cdot)\,\mathrm{d}x) &= \overline{b_i}(t,x,u(t,x)), \end{aligned} \qquad \text{"Nemytskii–type"}$$

where

$$\begin{aligned} \overline{a_{ij}} &\colon [0,\infty) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}, \\ \overline{b_i} &\colon [0,\infty) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \end{aligned}$$

are measurable functions.

2. The Nemytskii – case

Then the NLFPKE is

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}\left[\overline{a_{ij}}(t,x,u(t,x))u(t,x)\right] - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left[\,\overline{b_i}(t,x,u(t,x))u(t,x)\right],$$

$$u(0,\cdot) \text{ a given probability density on } \mathbb{R}^d,$$

its corresponding Kolmogorov operator is for $\varphi \in C_c^2(\mathbb{R}^d)$

$$L_{u(t,\cdot)}\varphi(t,x) = \frac{1}{2}\sum_{i,j=1}^{d}\overline{a_{ij}}(t,x,u(t,x))\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x) + \sum_{i=1}^{d}\overline{b_i}(t,x,u(t,x))\frac{\partial}{\partial x_i}\varphi(x),$$

and for $\sigma\sigma^T = (a_{ij})_{1\le i,j\le d}$ the corresponding DD (= McKean–Vlasov) SDE is

$$\begin{aligned} &\mathrm{d}X(t) = b(t,X(t),u(t,X(t)))\mathrm{d}t + \sigma(t,X(t),u(t,X(t)))\mathrm{d}W(t),\\ &\mathcal{L}_{X(t)}(\mathrm{d}x) = u(t,x)\mathrm{d}x, \quad t \ge 0. \end{aligned}$$

Note: Theorem I above still applies, if **Assumption** holds.

3. Perturbed porous media equation and nonlinear distorted Brownian motion

Ref.: [Barbu/R.: arXiv:1904.08291]
In this section we look at the following special Nemytskii–type NLFPKE

$$\begin{aligned} u_t - \tfrac{1}{2}\Delta\beta(u) + \operatorname{div}(Db(u)u) = 0 \text{ in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x),\ x\in\mathbb{R}^d, \end{aligned} \tag{pPME}$$

where $d \in \mathbb{N}$ and $\beta : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R} \to \mathbb{R}$, such that

(i) $\beta \in C^1(\mathbb{R}),\ \beta(0) = 0,\ \gamma \le \beta'(r) \le \gamma_1,\ \forall r \in \mathbb{R}$, for $0 < \gamma < \gamma_1 < \infty$.

(ii) $b \in C_b(\mathbb{R}) \cap C^1(\mathbb{R})$.

(iii) $D \in C_b(\mathbb{R}^d;\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$.

(iv) $D = -\nabla\Phi$, *where* $\Phi \in C^1(\mathbb{R}^d),\ \Phi \ge 1,\ \lim_{|x|_d\to\infty} \Phi(x) = +\infty$ *and there exists* $m \in (0,\infty)$ *such that* $\Phi^{-m} \in L^1(\mathbb{R}^d)$

(hence $\overline{a_{ij}}(t,x,r) := \frac{\beta(r)}{r}\delta_{ij}, \overline{b_i}(t,x,r) := b(r)D(x), r \in \mathbb{R}$).

3. Perturbed porous media equation and nonlinear distorted Brownian motion

Its corresponding Kolmogorov operator is

$$L_{u(t,x)} = \frac{1}{2}\frac{\beta(u(t,x))}{u(t,x)}\Delta - b(u(t,x))\langle \nabla\Phi, \nabla\cdot\rangle \quad \left\{ \begin{array}{l} \text{Generator of distorted} \\ \text{Brownian motion} \\ \text{if } \beta = id \text{ and } b = const.\ ! \end{array} \right.$$

and the corresponding DD (= McKean–Vlasov) SDE

$$\begin{aligned} \mathrm{d}X(t) &= -b(\mathcal{L}_{X(t)}(X(t)))\nabla\Phi(X(t))\mathrm{d}t + \sqrt{\frac{\beta(\mathcal{L}_{X(t)}(X(t)))}{\mathcal{L}_{X(t)}(X(t))}}\mathrm{d}W(t), \\ \mathcal{L}_{X(t)}(x) &:= \frac{\mathrm{d}\mathcal{L}_{X(t)}}{\mathrm{d}x}(x) = u(t,x), \quad t \geq 0. \qquad (NLDBM) \end{aligned}$$

"nonlinear distorted Brownian motion"

Remark

We shall see that by Theorem I and Theorem II below, the above DDSDE has a weak solution, so **"nonlinear distorted Brownian motion"** exists.

3. Perturbed porous media equation and nonlinear distorted Brownian motion

Remark

A typical example for Φ *as in* (iv) above is

$$\Phi(x) = C(1+|x|^2)^\alpha,\ x \in \mathbb{R}^d,$$

with $\alpha \in \left(0, \frac{1}{2}\right]$.

Now let us solve (pPME).
Consider the operator $A : D(A) \subset L^1 \to L^1$, defined by

$$\begin{aligned} Au &= -\Delta\beta(u) + \operatorname{div}(Db(u)u),\ \forall u \in D(A), \\ D(A) &= \{u \in L^1;\ -\Delta\beta(u) + \operatorname{div}(Db(u)u) \in L^1\}, \end{aligned}$$

in $L^1 = L^1(\mathbb{R}^d)$. Here, the differential operators Δ and div are taken in the sense of Schwartz distributions, i.e., in $\mathcal{D}'(\mathbb{R}^d)$. Obviously, the operator $(A, D(A))$ is closed on L^1. Denote by $\overline{D(A)}$ the closure of $D(A)$ in L^1.

3. Perturbed porous media equation and nonlinear distorted Brownian motion

Proposition I

Assume that hypotheses (i)–(iv) *hold. Then, the operator A is m-accretive, that is,*

$$R(I+\lambda A)=L^1,\ \forall\lambda>0,$$

$$|(I+\lambda A)^{-1}u-(I+\lambda A)^{-1}v|_1\le|u-v|_1,\ \forall\lambda>0,\ u,v\in L^1.$$

Furthermore,

$$\overline{D(A)}=L^1,$$

where "$\overline{}$" denotes the closure in L^1. Moreover, there exists $\lambda_0>0$ such that, for all $\lambda\in(0,\lambda_0)$,

$$\int_{\mathbb{R}^d}(I+\lambda A)^{-1}u_0\,dx=\int_{\mathbb{R}^d}u_0(x)dx,\ \forall u_0\in L^1,$$

$$(I+\lambda A)^{-1}u_0\ge 0,\quad a.e.\ in\ \mathbb{R}^d\ if\ u_0\ge 0,\ a.e.\ in\ \mathbb{R}^d.$$

3. Perturbed porous media equation and nonlinear distorted Brownian motion

Consider now the Cauchy problem associated with A, that is,

$$\frac{du}{dt} + Au = 0,\ t \geq 0, \qquad u(0) = u_0. \tag{CP}$$

A continuous function $u : [0, \infty) \to L^1$ is said to be a *mild solution to* (CP) if

$$u(t) = \lim_{h \to 0} u_h(t) \text{ in } L^1,\ \forall t \geq 0,$$

uniformly on compacts of $[0, \infty)$, where $u_h^0 = u_0$, and

$$u_h(t) = u_h^i,\ t \in [ih, (i+1)h),\ i = 0, 1, ...,$$

$$u_h^{i+1} + hAu_h^{i+1} = u_h^i,\ i = 0, ...$$

3. Perturbed porous media equation and nonlinear distorted Brownian motion

Since A is m-accretive, we have by the Crandall & Liggett theorem the following existence result for (CP).

Theorem II

Under Hypotheses (i)–(iv)*, there is a unique mild solution u to* (CP)*. Moreover, for every $u_0 \in \overline{D(A)} = L^1$, one has, for all $t \geq 0$,*

$$u(t) = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} u_0$$

uniformly on bounded intervals of $[0,\infty)$ in the strong topology in L^1. One also has that

$$\int_{\mathbb{R}^d} u(t,x)dx = \int_{\mathbb{R}^d} u_0(x)dx, \ \forall t \geq 0,$$

$$u(t,x) \geq 0, \ \text{a.e. on } (0,\infty)\times\mathbb{R}^d \text{ if } u_0 \geq 0, \ \text{a.e. in } \mathbb{R}^d.$$

The function u will be called the mild solution to (pPME)*.*
In particular, for each $t \geq 0$, $u(t,\cdot)$ is a probability density if so is u_0.

3. Perturbed porous media equation and nonlinear distorted Brownian motion

Theorem II (continued)

Furthermore, the map $t \to S(t)u_0$ is a continuous semigroup of (nonlinear) contractions on L^1, that is,

$$S(t)u_0 = u(t) = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} u_0 \ \ \text{``}= e^{tA}u_0\text{''},\ \forall t \geq 0,$$

$$S(t+s)u_0 = S(t)S(s)u_0,\ \forall t, s \geq 0,\ u_0 \in L^1,$$

$$\lim_{t\to 0} S(t)u_0 = u_0 \text{ in } L^1,$$

$$|S(t)u_0 - S(t)\bar{u}_0|_1 \leq |u_0 - \bar{u}_0|_1,\ \forall t \geq 0,\ u_0, \bar{u}_0 \in L^1.$$

In particular, $u(t,\cdot) = S(t)u_0,\ t \geq 0$, is a solution to (pPME) *in the sense of* (NLFPKE).

3. Perturbed porous media equation and nonlinear distorted Brownian motion

Consider the following subspace of L^1

$$\mathcal{M} \quad = \quad \left\{ u \in L^1; \int_{\mathbb{R}^d} \Phi(x)|u(x)|dx < \infty \right\}$$

with the norm

$$\|u\| = \int_{\mathbb{R}^d} \Phi(x)|u(x)|dx, \ \forall u \in \mathcal{M}.$$

It turns out that the semigroup $S(t)$ leaves $\mathcal{M}$ invariant. More precisely,

Proposition II

Assume that (i)–(iv) *hold. Then*

$$\|S(t)u_0\| \leq \|u_0\| + \rho t |u_0|_1, \ \forall u_0 \in \mathcal{M},$$

where $\rho = (m+1)|\Delta\Phi|_\infty \gamma_1$.

3. Perturbed porous media equation and nonlinear distorted Brownian motion

Theorem I + II $\Longrightarrow$

Theorem III

There exists a probabilistically weak solution to (NLDBM)*.*
Furthermore, by [Barbu/R.: SPDE Analysis and Comp. 2020+] it is unique in law provided $u_0 \in L^1 \cap L^\infty$*.*

4. Asymptotic behaviour and unique stationary solution: The H–Theorem

Additionally to (i)-(iv), assume

(v) $b(r) \geq b_0 > 0 \quad \forall r \geq 0$.

Define $\eta \in C([0,\infty)) \cap C^2((0,\infty))$ by

$$\eta(r) := -\int_0^r d\tau \int_\tau^1 \frac{\beta'(s)}{sb(s)}\, ds, \ \forall r \geq 0,$$

and define the function $V : D(V) = \{u \in \mathcal{M};\ u \geq 0, \text{ a.e. on } \mathbb{R}^d\} \to \mathbb{R}$ (which will turn out to be a Lyapunov function for $S(t)u_0,\ t \geq 0$, in the sense of [Pazy 1981])

$$V(u) := \int_{\mathbb{R}^d} \eta(u(x))dx + \int_{\mathbb{R}^d} \Phi(x)u(x)dx = -S[u] + E[u].$$

Since, by (i), (iv),

$$\frac{\gamma}{r|b|_\infty} \leq \frac{\beta'(r)}{rb(r)} \leq \frac{\gamma_1}{rb_0}, \ \forall r \geq 0,$$

we have

$$\frac{\gamma_1}{b_0}\, 1_{[0,1]}(r) r(\log r - 1) + \frac{\gamma}{|b|_\infty}\, 1_{(1,\infty)}(r) r(\log r - 1) \leq \eta(r)$$
$$\leq \frac{\gamma}{|b|_\infty}\, 1_{[0,1]}(r) r(\log r - 1) + \frac{\gamma_1}{b_0}\, 1_{(1,\infty)}(r) r(\log r - 1).$$

4. Asymptotic behaviour and unique stationary solution: The H–Theorem

Furthermore, exactly as in [Jordan/Kinderlehrer/Otto 1998], one proves that $(u \ln u)^- \in L^1$ if $u \in D(V)$. Hence $S[u]$ is well-defined and $V(u) \in (-\infty, \infty]$ for all $u \in D(V)$. We define $V = \infty$ on $L^1 \setminus D(V)$. Then, obviously, $V : L^1 \to (-\infty, \infty]$ is convex and L^1_{loc}-lower semicontinuous on $\mathcal{M}$-balls. $S[u]$ is called in the literature the entropy of the system, while $E[u]$ is the mean field energy.
In fact, according to the general theory of thermostatics, the functional $S = S[u]$ is a generalized entropy because its kernel $-\eta$ is a strictly concave continuous functions on $(0, \infty)$ and $\lim\limits_{r \downarrow 0} \eta'(r) = +\infty$. In the special case $\beta(s) \equiv s$ and $b(s) \equiv 1$, $\eta(r) \equiv r(\log r - 1)$ and so $S[u]$ reduces to the classical Boltzman-Gibbs entropy.
Define $\Psi : D(\Psi) \subset L^1 \to [0, \infty)$ by

$$\Psi(u) = \int_{\mathbb{R}^d} \left| \frac{\beta'(u)\nabla u}{\sqrt{ub(u)}} - D\sqrt{ub(u)} \right|_d^2 dx,$$

$$D(\Psi) = \{u \in L^1 \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d); u \geq 0,\ \Psi(u) < \infty\},$$

and $\Psi = \infty$ on $L^1 \setminus D(\Psi)$. Then Ψ is L^1_{loc}-lower semicontinuous on L^1-balls. For $u_0 \in L^1, u_0 \geq 0$, define the *ω-limit set*

$$\omega(u_0) := \{\lim_{n \to \infty} S(t_n)u_0 \text{ in } L^1_{\mathrm{loc}} \text{ for some } t_n \to \infty\}.$$

4. Asymptotic behaviour and unique stationary solution: The H–Theorem

Theorem IV ("H-Theorem", Barbu/R.: arXiv:1904.08291)

Part (a):
Assume that hypotheses (i)–(v) *hold. Then the function V defined above is a Lyapunov function for $S(t)$, $t \geq 0$, that is,*

$$S(t)u_0 \in D(V),\ \forall t \geq 0, u_0 \in D_0(V) := D(V) \cap \{V < \infty\} \text{ and}$$
$$V(S(t)u_0) \leq V(S(s)u_0),\ \forall u_0 \in D_0(V), 0 \leq s \leq t < \infty.$$

Moreover, we have, for all $u_0 \in D_0(V)$,

$$V(S(t)u_0) + \int_s^t \Psi(S(\sigma)u_0)d\sigma \leq V(S(s)u_0) \text{ for } 0 \leq s \leq t < \infty,$$
$$\exists\ u_\infty \in \omega(u_0) \cap \{u \in D(\Psi); \Psi(u) = 0\} \quad (\subset L^1).$$

For any such u_∞ we have either $u_\infty \equiv 0$ or $u_\infty > 0$ a.e. In the latter case there exists $\mu \in \mathbb{R}$ such that

$$u_\infty = g^{-1}(-\Phi + \mu), \text{ where } g(r) = \int_1^r \frac{\beta'(s)}{sb(s)}\,ds,\ r > 0.$$

4. Asymptotic behaviour and unique stationary solution: The H–Theorem

Remark

Before we come to part (b) of Theorem IV we note that

$$\Psi(u_\infty) = \int_{\mathbb{R}^d} \left| \frac{\beta'(u_\infty)\nabla u_\infty}{\sqrt{u_\infty b(u_\infty)}} + \nabla\Phi\sqrt{u_\infty b(u_\infty)} \right|_{\mathbb{R}^d}^2 dx = 0$$

$$\iff \quad \frac{\beta'(u_\infty)\nabla u_\infty}{u_\infty b(u_\infty)} = -\nabla\Phi \quad dx - a.e.$$

$$\frac{\beta'(u_\infty)\nabla u_\infty}{u_\infty b(u_\infty)} = \nabla\left(\int_1^{u_\infty} \frac{\beta'(s)}{sb(s)}\,ds\right) = \nabla g(u_\infty)$$

$$\iff \quad g(u_\infty) = -\Phi + \mu \qquad \text{for some } \mu \in \mathbb{R}.$$

$$\iff \quad u_\infty = g^{-1}(-\Phi + \mu)$$

4. Asymptotic behaviour and unique stationary solution: The H–Theorem

Theorem IV ("H-Theorem")

Part (b):
Assume in addition to (i) - (v)

(vi) $\gamma_1 \Delta \Phi - b_0 |\nabla \Phi|^2 \leq 0$.

Let $u_0 \in D_0(V) \setminus \{0\}$. Set

$$\tilde{\omega}(u_0) = \{\lim_{n \to \infty} S(t_n) u_0 \text{ in } L^1, \{t_n\} \to \infty\}.$$

Then

$$\omega(u_0) = \tilde{\omega}(u_0) = \{u_\infty\}, \tag{$\star$}$$

$u_\infty > 0$ a.e., $u_\infty \in D_0(V) \cap D(\Psi), \Psi(u_\infty) = 0$, $|u_\infty|_1 = |u_0|$, and it is given by

$$u_\infty(x) = g^{-1}(-\Phi(x) + \mu), \ \forall x \in \mathbb{R}^d,$$

where μ is the unique number in $\mathbb{R}$ such that

$$\int_{\mathbb{R}^d} g^{-1}(\Phi(x) + \mu) dx = \int_{\mathbb{R}^d} u_0 dx.$$

4. Asymptotic behaviour and unique stationary solution: The H–Theorem

Theorem IV ("H-Theorem" continued)

In particular, for all $u_0 \in D_0(V)$ *with the same* L^1*-norm the sets in* $(\star)$ *coincide. Furthermore,* u_∞ *is the unique element in* $D_0(V)$ *such that* $S(t)u_\infty = u_\infty$ *for all* $t \geq 0$*. In particular, in* $D_0(V)$ *equation* (pPME) *has a unique stationary probability solution which is the unique invariant measure for our probabilistically weak solution to (NLDBM), and so an invariant measure for the* **"nonlinear distorted Brownian motion"**.

Remark

Typical examples for Φ *satisfying* (iv) and (vi) are Φ as in (iv) such that $\Phi = \text{const.}$ (≥ 1) on a ball of radius R_1 around zero and Φ behaves like $C(1 + |x|^2)^\alpha$, $\alpha \in (0, \frac{1}{2}]$, outside a ball around zero of radius $R_2 > R_1$, where R_1 and R_2 are properly chosen depending on γ_1 and b_0.