The vanishing discount problem for systems of Hamilton-Jacobi equations

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Vanishing discount problem

Scalar Case: We consider the Hamilton-Jacobi equation

$$(\mathrm{P}_\lambda) \qquad \lambda v(x) + H(x, Dv(x)) = 0 \quad \text{in } \mathbb{T}^n.$$

Here

$$\begin{cases} v = v^\lambda \ \text{ the unknown function on } \mathbb{T}^n, \\ Dv = (v_{x_1}, \dots, v_{x_n}), \\ \lambda > 0 \ \text{ a given constant, discount factor,} \\ H \ \text{ a given function of } (x, p) = (x, Dv(x)). \end{cases}$$

Problem: asymptotic behavior of v^λ as $\lambda \to 0$.

Convex, coercive HJ equations

Hypotheses:

(H0) Continuity: $H \in C(\mathbb{T}^n \times \mathbb{R}^n)$.

(H1) H is convex,

$$p \mapsto H(x,p) \text{ is convex.}$$

(H2) H is coercive,

$$\lim_{|p|\to\infty} \min_{x\in\mathbb{T}^n} H(x,p) = \infty.$$

Property of H:

$$H(x,p) \geq \delta|p| - C \quad (\exists\delta > 0,\ \exists C > 0).$$

Example: $H(x,p) = |p|^m - f(x)$, $m \geq 1$, $f \in C(\mathbb{T}^n)$.

Theorem 1 For each $\lambda > 0$ problem (P$_\lambda$) has a unique solution v^λ. Furthermore,

$$(\lambda v^\lambda)_{\lambda>0} \quad \text{is uniformly bounded,}$$
$$(v^\lambda)_{\lambda>0} \quad \text{is equi-Lipschitz continuous.}$$

- If $C_0 \geq |H(x,0)|$, then

$$\lambda(C_0/\lambda) + H(x,0) \geq 0, \qquad \lambda(-C_0/\lambda) + H(x,0) \leq 0,$$

and, by comparison, $-C_0/\lambda \leq v^\lambda(x) \leq C_0/\lambda.$

- Since $H(x,p) \geq \delta|p| - C$, we have

$$\delta|Dv^\lambda(x)| \leq C + \lambda\|v^\lambda\|_\infty.$$

Notation. Lagrangian of H:

$$L(x,\xi) := \sup_{p\in\mathbb{R}^n} [\xi \cdot p - H(x,p)].$$

Properties: L is convex and lower semicontinuous on $\mathbb{T}^n \times \mathbb{R}^n$.

$$\begin{aligned} L(x,\xi) &\geq -H(x,0), \\ L(x,\xi) &\geq A|\xi| - H(x, A\xi/|\xi|) \\ &\geq A|\xi| - \max_{|p|\leq A} H(x,p) \quad \forall A > 0, \\ L(x,\xi) &\leq \sup_p (|\xi||p| - \delta|p| + C) = C \quad \forall \xi \in B_\delta. \end{aligned}$$

Recall here that $H(x,p) \geq \delta|p| - C$.

Ergodic problem

Formal expansion of the solution of (P_λ):

$$v^\lambda(x) \approx a_0(x)\lambda^{-1} + a_1(x) + a_2(x)\lambda + \cdots .$$

Plug this into (P_λ):

$$a_0(x) + a_1(x)\lambda + a_2(x)\lambda^2 + \cdots$$
$$+ H(x, Da_0(x)\lambda^{-1} + Da_1(x) + Da_2(x)\lambda + \cdots) \approx 0.$$

We deduce that

$$Da_0(x) = 0 \quad \text{i.e.} \quad a_0(x) \equiv a_0 \text{ (constant)},$$
$$a_0 + H(x, Da_1(x)) = 0.$$

The ergodic problem or additive eigenvalue problem:

The problem of finding a constant $c \in \mathbb{R}$ and a function $u \in C(\mathbb{T}^n)$ satisfying

$$\text{(E)} \qquad H(x, Du(x)) = c \quad \text{in } \mathbb{T}^n.$$

A classical result:

Theorem 2 (Lions-Papanicolaou-Varadhan, 1987)

Under (H0), (H2), there exists a solution $(c, u) \in \mathbb{R} \times C(\mathbb{T}^n)$ of (E). Moreover, the constant c is unique.

- The constant c is called the critical value, additive eigenvalue, or ergodic constant.

Their proof is to show that for some $(c, u) \in \mathbb{R} \times C(\mathbb{T}^n)$,

$$\begin{cases} -\lambda v^\lambda(x) \quad \to \quad c \quad \text{uniformly on } \mathbb{T}^n, \\ v^\lambda(x) + \lambda^{-1} c \quad \to \quad u(x) \quad \text{uniformly on } \mathbb{T}^n \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{along a subsequence}, \end{cases}$$

Main question: does the whole family $\{v^\lambda + \lambda^{-1}c\}_{\lambda>0}$ converges to a function as $\lambda \to 0+$?

• The ergodic problem (E) has multiple solutions. If u is a solution of (E), then $u +$ const is a solution. Consider the case

$$Du \cdot (Du - D\psi) = 0 \quad \text{in } \mathbb{T}^n, \quad \text{with } \psi \in C^1(\mathbb{T}^n).$$

We have many solutions:

$$u = C_1, \qquad u = \psi + C_2, \qquad u = \min\{C_1, \psi + C_2\}.$$

• Ergodic problem (E) arises in the ergodic optimal control, the homogenization of HJ equations, and the large-time behavior of solutions of evolutionary HJ equations.
A decisive result on the main question:

Theorem 3 (Davini-Fathi-Iturriaga-Zavidovique, 2016)

Assume (H0)–(H2). Let c be the critical value. Then, for some function $v^0 \in C(\mathbb{T}^n)$, as $\lambda \to 0+$,

$$v^\lambda(x) + \lambda^{-1}c \quad \to \quad v^0(x) \quad \text{in } C(\mathbb{T}^n).$$

• If H is not convex, the convergence of the whole family does not hold in general. A counterexample by B. Ziliotto (2019).

Related work:

1) A. Davini, A. Fathi, R. Iturriaga, M. Zavidovique,

Coercive, convex HJ equation on $\mathbb{T}^n$ (closed manifold).

2) E. S. Al-Aidarous, E. O. Alzahrani, HI, A. M. M. Younas,

Coercive, convex HJ equation on a bounded domain with the Neumann type BC.

3) H. Mitake, H. V. Tran

Viscous HJ equation on $\mathbb{T}^n$, with coercive and convex Hamiltonian. (2nd-order degenerate elliptic PDEs.)

4) D. Gomes, H. Mitake, H. V. Tran

Coercive, quasi-convex HJ equation on $\mathbb{T}^n$.

5) HI, H. Mitake, H. V. Tran,

2nd-order fully nonlinear, convex, degenerate elliptic PDEs on $\mathbb{T}^n$ or on a bounded domain with BC.

6) B. Ziliotto,

A counterexample, with non-convex Hamiltonian.

- Use of Mather measures.

An approach to Theorem 3

We review the proof of Theorem 3 (Davini et al.).

$\mathcal{P} = \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n)$ all Borel probability measures on $\mathbb{T}^n \times \mathbb{R}^n$.

$\mathcal{P}_1 = \mathcal{P}_1(\mathbb{T}^n \times \mathbb{R}^n)$ all $\mu \in \mathcal{P}$ such that

$$\langle \mu, |\xi| \rangle := \int_{\mathbb{T}^n \times \mathbb{R}^n} |\xi| \, \mu(dx d\xi) < \infty.$$

(the function $(x, \xi) \mapsto |\xi|$ is denoted by $|\xi|$)

Fix $(z, \lambda) \in \mathbb{T}^n \times [0, \infty)$.

$\mathfrak{C}(z, \lambda)$ (closed measures)

$$:= \{\mu \in \mathcal{P}_1 \mid \lambda\psi(z) = \langle \mu, \xi \cdot D\psi + \lambda\psi \rangle \ \ \forall \psi \in C^1(\mathbb{T}^n)\}.$$

Note that

$$\lambda u(x) + H(x, Du(x)) = \sup_{\xi} (\lambda u(x) + \xi \cdot Du(x) - L(x, \xi)).$$

When $\lambda = 0$, the defining condition reads

$$0 = \langle \mu, \xi \cdot D\psi \rangle \quad \forall \psi \in C^1(\mathbb{T}^n).$$

So, we write $\mathfrak{C}(0)$ for $\mathfrak{C}(z, 0)$.

Theorem 4 Assume (H0)–(H2). If $\lambda > 0$, then

$$\lambda v^\lambda(z) = \min_{\mu \in \mathfrak{C}(z,\lambda)} \langle \mu, L \rangle.$$

- Any minimizer μ of the optimization problem above is called a discounted Mather measure. $\mathfrak{M}(z, \lambda) = \mathfrak{M}(z, \lambda, L)$.

Theorem 5 Assume (H0)–(H2). Let c be the critical value. Then

$$-c = \min_{\mu \in \mathfrak{C}(0)} \langle \mu, L \rangle.$$

- Any minimizer μ of the optimization problem

$$\min_{\mu \in \mathfrak{C}(0)} \langle \mu, L \rangle.$$

is called a Mather measure. $\mathfrak{M} = \mathfrak{M}(L)$.

- We assume henceforth that $c = 0$. (Replace H by $H - c$ if needed.)

The family $(v^\lambda)_{\lambda > 0}$ is equi-Lipschitz and uniformly bounded on $\mathbb{T}^n$ ($\Rightarrow$ relatively compact in $C(\mathbb{T}^n)$ by A^2 theorem).

(Uniform boundedness) Let $v_0 \in C(\mathbb{T}^n)$ be a solution of (E). Let $C > 0$ be a constant such that $\|v_0\|_\infty \le C$, and note that $v_0 + C$ (reps. $v_0 - C$) is a supersolution (resp. a subsolution) of (P$_\lambda$).
By the comparison theorem, which is valid for (P$_\lambda$) with $\lambda > 0$,

$$v_0 - C \le v^\lambda \le v_0 + C \quad \forall \lambda > 0.$$

$\mathcal{V}$ all accumulation points of $(v^\lambda)_{\lambda>0}$ in $C(\mathbb{T}^n)$ as $\lambda \to 0+$. By the observation above, $\mathcal{V} \neq \emptyset$.

To show Theorem 3 (Davini et al.), it is enough to prove that $\#(\mathcal{V}) \leq 1$.

The main part of the proof (Theorem 3):

(Claim 1) $\langle \mu, v \rangle \leq 0 \qquad \forall v \in \mathcal{V}, \ \forall \mu \in \mathfrak{M}.$

(Claim 2) For $\forall v, w \in \mathcal{V}$, $\forall z \in \mathbb{T}^n$, $\exists \mu \in \mathfrak{M}$ s.t.

$$w(z) \leq v(z) + \langle \mu, w \rangle.$$

Claims 1 and 2 show that $v, w \in \mathcal{V} \Rightarrow v = w$. I.e., $\#\mathcal{V} \leq 1$.

Proof (sketch) of Claims 1 and 2

Davini et al. have obtained two representations of the limit function of (v^λ). Here is one of them.

Theorem 6 Assume (H0)–(H2) and that $c = 0$. Let $v^0 \in C(\mathbb{T}^n)$ be the limit function of (v^λ), that is,

$$v^0 = \lim_{\lambda \to 0+} v^\lambda \quad \text{in } C(\mathbb{T}^n).$$

Then

$$v^0(x) = \max\{w(x) \mid w \in \mathcal{S},\ \langle \mu, w \rangle \le 0\ \forall \mu \in \mathfrak{M}\},$$

where $\mathcal{S}$ denotes the set of all solutions of (E).

Remarks. • Davini et al. have proved Theorem 4 by using techniques from optimal control or dynamical systems (value functions, the Hopf-Lax-Oleinik formula).
Mitake-Tran use the adjoint method introduced by L. C. Evans.
Mitake-Tran-HI use the convex duality argument similar to those used by Gomes (Duality principles for fully nonlinear elliptic equations, 2005) and Mikami-Thieullen (Duality theorem for the stochastic optimal control problem, 2006). A feature of this approach by Mitake-Tran-HI is that it belongs to functional analysis and is easily adopted to different situations, for instance, 2nd-order elliptic equations, nonlocal equations, systems of PDEs without going into detailed studies of the underlying dynamics.
Siconolfi-HI use the convex duality in the form of the Hahn-Banach theorem.

• The measures $\mu \in \bigcup_{z,\lambda} \mathfrak{M}(z, \lambda, L)$ are supported in a common compact subset of $\mathbb{T}^n \times \mathbb{R}^n$. This is a consequence of the fact that $\sup_{\lambda>0} \|Dv^\lambda\|_\infty < \infty$ (equi-Lipschitz).
The set $\bigcup_{z,\lambda} \mathfrak{M}(z, \lambda, L)$ is relatively compact in the topology of the weak convergence in the sense of measures.

Systems of HJ equations

Some recent results with Liang Jin.

The problem is now the m-system

$$\begin{cases} \lambda v_1^\lambda + H_1(x, Dv_1^\lambda, v^\lambda) = 0 \quad \text{in } \mathbb{T}^n, \\ \qquad\qquad \vdots \\ \lambda v_m^\lambda + H_m(x, Dv_m^\lambda, v^\lambda) = 0 \quad \text{in } \mathbb{T}^n. \end{cases}$$

We write for the system above simply

$$(\mathrm{P}_\lambda) \qquad \lambda v^\lambda + H(x, Dv^\lambda, v^\lambda) = 0 \quad \text{in } \mathbb{T}^n,$$

where $v^\lambda = (v_i^\lambda)$ and $H = (H_i)$.

Assume

(1) $H_i \in C(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m)$.

(2) H_i is coercive, that is,

$$\lim_{|p|\to\infty} H_i(x,p,u) = \infty \text{ uniformly for } (x,u) \in \mathbb{T}^n \times B_R^m, \ \forall R > 0.$$

(3) $(p,u) \mapsto H_i(x,p,u)$ is convex for any $x \in \mathbb{T}^n$.

(4) $H = (H_i)$ is monotone, that is, for $u, v \in \mathbb{R}^m$,

$$(u-v)_k = \max_i (u-v)_i \geq 0 \implies H_k(x,p,u) \geq H_k(x,p,v).$$

(5) $H(x, Du, u) = 0$ has a solution $u \in C(\mathbb{T}^n)^m$.

Theorem 7 Assume (1)–(5) above. Then, as $\lambda \to 0+$, we have

$$v^\lambda \to v^0 \quad \text{in } C(\mathbb{T}^n)^m$$

for some $v^0 \in C(\mathbb{T}^n)^m$.

Davini-Zavidovique (2019) have studied the case where the coupling is linear and the coupling coefficients are constants.

Examples (coupling)

(E1)
$$\begin{cases} \lambda u_1 + |Du_1| + u_1 - u_2 = f_1(x), \\ \lambda u_2 + |Du_2|^2 + u_2 - u_1 = f_2(x). \end{cases}$$

(E2)
$$\begin{cases} \lambda u_1 + |Du_1| + (u_1 - u_2)^+ = f_1(x), \\ \lambda u_2 + |Du_2| + (u_2 - u_1)^+ = f_2(x). \end{cases}$$

(E3)
$$\begin{cases} \lambda u_1 + |Du_1| + u_1 = f_1(x), \\ \lambda u_2 + |Du_2|^2 + u_2 = f_2(x). \end{cases}$$

Some ideas for the proof.

- Set $\mathbb{I} = \{1, \ldots, m\}$ and

$$\begin{aligned} L_i(x,\xi,\eta) &= \sup_{(p,u)} [\xi \cdot p + \eta \cdot u - H_i(x,p,u)], \\ Y_i &= \{\eta \in \mathbb{R}^m \mid \sum_{j \in \mathbb{I}} \eta_j \geq 0,\ \eta_j \leq 0 \text{ for } j \neq i\}. \end{aligned}$$

Theorem 8 Assume (1)–(3). Then,

$$H \text{ monotone} \iff L_i(x,\xi,\eta) = \infty \quad \text{for } \eta \in \mathbb{R}^m \setminus Y_i$$

• When $\lambda > 0$, we set $T^\lambda(\eta) = 1 + \lambda^{-1} \sum_j \eta_j$ for $\eta \in \mathbb{R}^m$.

Note that

$$T^\lambda(\eta) \geq 1 \quad \forall \eta \in Y_i,\ i \in \mathbb{I},$$

$$H^\lambda_{\phi+\lambda T^\lambda \mathbf{1}}(x, D(u+\mathbf{1}), u+\mathbf{1}) = H^\lambda_\phi(x, Du, u),$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^m$ and

$$H^\lambda_\phi(x, pu) = \left(\lambda u_i + \sup_{(\xi,\eta)}(\xi \cdot + \eta \cdot u - \phi_i(x, \xi, \eta))\right)_{i \in \mathbb{I}}.$$

$\mathcal{P}(\lambda)$ the set of collections $\mu = (\mu_i)_{i \in \mathbb{I}}$ of nonnegative Borel measures μ_i on $\mathbb{T}^n \times \mathbb{R}^n \times Y_i$ such that

$$\langle \mu_i, |\xi| + |\eta| \rangle < \infty \quad \forall i \in \mathbb{I} \quad \text{and} \quad \sum_{i \in \mathbb{I}} \langle \mu_i, T^\lambda \rangle = 1.$$

$\mathcal{P}(0)$ the set of collections $\mu = (\mu_i)$ of nonnegative Borel measures μ_i on $\mathbb{T}^n \times \mathbb{R}^n \times Y_i$ such that

$$\langle \mu_i, |\xi| + |\eta| \rangle < \infty \quad \text{and} \quad \sum_{i \in \mathbb{I}} \langle \mu_i, 1 \rangle \leq 1.$$

• Fix $(z, k, \lambda) \in \mathbb{T}^n \times I \times [0, \infty)$.

$\mathfrak{C}(z, k, \lambda)$, closed measures all $\mu = (\mu_i) \in \mathcal{P}(\lambda)$ such that

$$\lambda \psi_k(z) = \sum_{i \in \mathbb{I}} \langle \mu_i, \xi \cdot D\psi_i + \eta \cdot \psi + \lambda \psi_i \rangle \quad \forall \psi \in C^1(\mathbb{T}^n)^m.$$

Theorem 9 Assume (1)–(4). Then, if $\lambda > 0$,

$$\lambda v_k^\lambda(z) = \min_{\mu \in \mathfrak{C}(z,k,\lambda)} \sum_{i \in I} \langle \mu_i, L_i \rangle.$$

Discounted Mather measures $\mathfrak{M}(z, k, \lambda)$.

Proof (sketch). We have $\|(v^\lambda, Dv^\lambda)\|_\infty < \infty$, We may assume that for some $R > 0$,

$$\begin{cases} L_i(x, \xi, \eta) = +\infty & \text{if } (\xi, \eta) \notin K_i, \\ L_i \in C(\mathbb{T}^n \times K_i), \end{cases}$$

where

$$K_i = \overline{B}_R^n \times (\overline{B}_R^m \cap Y_i), \quad i \in \mathbb{I}.$$

$\mathcal{F}(\lambda)$ all pairs $u = (u_i)_{i\in\mathbb{I}} \in C(\mathbb{T}^n)^m$ and $\phi = (\phi_i)_{i\in\mathbb{I}} \in \prod_{i\in\mathbb{I}} C(\mathbb{T}^n \times K_i)$ such that

$$\lambda u(x) + H_\phi(x, Du(x), u(x)) \leq 0 \quad \text{in } \mathbb{T}^n,$$

where $H_\phi = (H_{\phi,i})_{i\in\mathbb{I}}$ and

$$H_{\phi,i}(x,p,v) = \max_{(\xi,\eta)\in K_i} [p\cdot\xi + v\cdot\eta - \phi_i(x,\xi,\eta)].$$

Our claim now is: Theorem 9 holds when we replace $\mathfrak{C}(z,k,\lambda)$ by

$$\mathfrak{C}_K(z,k,\lambda) := \{\mu = (\mu_i) \in \mathfrak{C}(z,k,\lambda) \mid \operatorname{supp}\mu_i \subset \mathbb{T}^n \times K_i\}.$$

Similarly, $\mathcal{P}_K(\lambda)$ for $\lambda \geq 0$.

Set

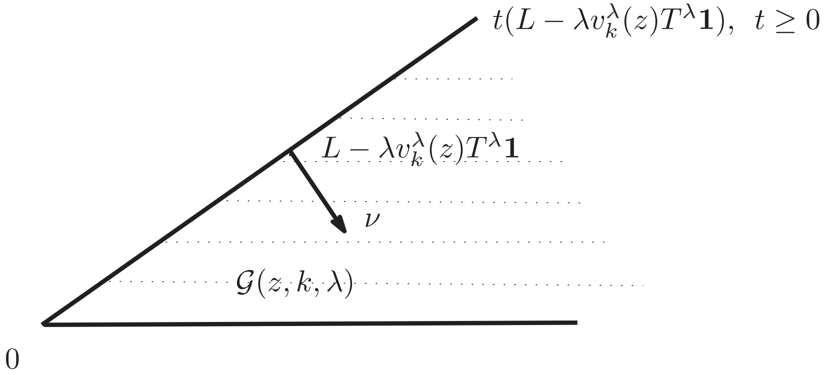
$$\mathcal{G}(z,k,\lambda) = \{\phi - \lambda u_k(z) T^{\lambda}\mathbf{1} \mid (u,\phi) \in \mathcal{F}(\lambda)\},$$

where $\mathbf{1} = (1,\ldots,1) \in \mathbb{R}^m$.
This is a closed convex cone in $\prod_{i\in\mathbb{I}} C(\mathbb{T}^n \times K_i)$ with vertex at the origin.

Theorem 10 Let $(z,k,\lambda) \in \mathbb{T}^n \times \mathbb{I} \times (0,\infty)$ and $\mu \in \mathcal{P}_K(\lambda)$. Then, $\mu \in \mathfrak{C}_K(z,k,\lambda)$ if and only if

$$\sum_{i\in\mathbb{I}} \langle \mu_i, g_i\rangle \geq 0 \quad \forall g = (g_i) \in \mathcal{G}(z,k,\lambda).$$

Proof (pictorial) $(\exists \boldsymbol{\nu} \in \mathfrak{M}(\boldsymbol{z}, \boldsymbol{k}, \boldsymbol{\lambda}))$



$\prod_{\boldsymbol{i} \in \mathbb{I}} \boldsymbol{C}(\mathbb{T}^{\boldsymbol{n}} \times \boldsymbol{K_i})$

Thank you for your attention!

Appendix

Theorem 11 Let χ, $u \in C(\mathbb{T}^n)$. Let $(z, \lambda) \in \mathbb{T}^n \times [0, \infty)$. Assume (H0)–(H2) and that u is a subsolution of $\lambda u + H(x, Du) = \chi$ in $\mathbb{T}^n$. Then

$$\lambda u(z) \leq \langle \mu, L + \chi \rangle \quad \forall \mu \in \mathfrak{C}(z, \lambda).$$

Proof (sketch). Assume that $u \in C^1$. Then

$$\lambda u(x) + \xi \cdot Du(x) \leq L(x, \xi) + \chi(x),$$

which implies

$$\begin{aligned} \lambda u(z) &= \langle \mu, \lambda u + \xi \cdot Du \rangle \quad (\because \ \mu \in \mathfrak{C}(z, \lambda)) \\ &\leq \langle \mu, L + \chi \rangle \ \forall \mu \in \mathfrak{C}(z, \lambda). \qquad \square \end{aligned}$$

Claim 1: Let $v \in \mathcal{V}$ and $\mu \in \mathfrak{M}$. If we set $\chi := -\lambda v^{\lambda}$, then

$$H(x, Dv^{\lambda}) = \chi \quad \text{in } \mathbb{T}^n,$$

and, by Theorem 11,

$$\begin{aligned} 0 &\leq \langle \mu, L + \chi \rangle = \langle \mu, L - \lambda v^{\lambda} \rangle \\ &= \underbrace{\langle \mu, L \rangle}_{=0} - \langle \mu, \lambda v^{\lambda} \rangle = -\lambda \langle \mu, v^{\lambda} \rangle, \end{aligned}$$

and

$$\langle \mu, v^{\lambda} \rangle \leq 0.$$

In the limit as $\lambda \to 0+$, we get Claim 1.

Claim 2: Fix any $v, w \in \mathcal{V}$ and $z \in \mathbb{T}^n$. Choose a sequence $\lambda_j \to 0+$ such that

$$v^{\lambda_j} \to v \quad \text{in } C(\mathbb{T}^n).$$

By Theorem 4, we may choose a discounted Mather measure $\mu_j \in \mathfrak{M}(z, \lambda_j)$. Observe that

$$\lambda_j w + H(x, Dw) = \lambda_j w,$$

and, by Theorem 11,

$$\lambda_j w(z) \le \langle \mu_j, L + \lambda_j w\rangle = \underbrace{\langle \mu_j, L\rangle}_{=\lambda_j v^{\lambda_j}(z)} + \lambda_j \langle \mu_j, w\rangle$$

$$= \lambda_j v^{\lambda_j}(z) + \lambda_j \langle \mu_j, w\rangle.$$

Dividing the above by λ_j and taking the limit along a subsequence of (λ_j), we get

$$w(z) \leq v(z) + \langle \mu, w \rangle$$

for some $\mu \in \mathfrak{M}$ and, hence, $w(z) \leq v(z)$.

- Since $(v^\lambda, L) \in \mathcal{F}(\lambda)$, we have $L - \lambda v_k^\lambda(z)T^\lambda \mathbf{1} \in \mathcal{G}(z,k,\lambda)$ and, for all $\mu \in \mathfrak{C}(z,k,\lambda)$,

$$0 \leq \sum_{i \in \mathbb{I}} \langle \mu_i, L_i - \lambda v_k^\lambda(z)T^\lambda \rangle = -\lambda v_k^\lambda(z) + \sum_{i \in \mathbb{I}} \langle \mu_i, L_i \rangle.$$

- $\exists \nu \in \mathfrak{C}(z,k,\lambda)$ minimizer: Note that if $\|\phi\|_\infty < 1$, then $(v^\lambda, L + \mathbf{1} + \phi) \in \mathcal{F}(\lambda)$. This implies that $\operatorname{int} \mathcal{G}(z,k,\lambda) \neq \emptyset$. We may show that $L - \lambda v_k^\lambda(z)T^\lambda \mathbf{1} \in \partial \mathcal{G}(z,k,\lambda)$ By the Hahn-Banach theorem, $\exists \nu \in \left(\prod_{i \in \mathbb{I}} C(K_i)\right)^*$ such that $\nu \neq 0$ and

$$\langle \nu, L - \lambda v_k^\lambda(z)T^\lambda \mathbf{1} \rangle \leq \langle \nu, g \rangle \quad \forall g \in \mathcal{G}(z,k,\lambda).$$

Since $t(L - \lambda v_k^\lambda(z)T^\lambda \mathbf{1}) \in \mathcal{G}(z,k,\lambda)$, we see that

$$\langle \nu, L - \lambda v_k^\lambda(z)T^\lambda \mathbf{1} \rangle = 0.$$

- For $\phi = (\phi_i)$, if $\phi_i \geq 0 \ \forall i \in \mathbb{I}$, then $(v^\lambda, L + \phi) \in \mathcal{F}(\lambda)$. This, with the Riesz theorem, implies that $\nu_i \geq 0$ and are Radon measures.