Some recent results on multilinear pseudo-differential operators

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In this lecture, I will survey some recent results on bilinear Fourier multiplier operators. Although most of the results are extended to the case of multilinear pseudo-differential operators, I will restrict to the bilinear case and to the case of Fourier multiplier operators.

The topics will be
1. Bilinear Fourier multipliers of Hörmander-Mihlin type
2. Multipliers of exotic class
3. Generalization of the bilinear exotic class

Some of the results are based on my joint works with Naohito Tomita (Osaka Univ.) and Tomoya Kato (Gunma Univ.).

1. Bilinear Fourier multipliers of Hörmander-Mihlin type

1-1. Bilinear Fourier multiplier operators

For $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and for $f, g \in \mathcal{S}(\mathbb{R}^n)$, we define

$$\begin{aligned} T_\sigma(f,g)(x) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{2\pi i x\cdot(\xi+\eta)} \sigma(\xi,\eta) \widehat{f}(\xi) \widehat{g}(\eta) \, d\xi d\eta \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x-y, x-z) f(y) g(z) \, dy dz, \quad x \in \mathbb{R}^n, \end{aligned}$$

where $\wedge$ denotes the Fourier transform and K is the inverse Fourier transform of σ. The operator T_σ is called the **bilinear Fourier multiplier operator** and the function σ is called the **multiplier**.

If X, Y, Z are function spaces on $\mathbb{R}^n$ equipped with quasi-norms, and if there exists a constant A such that

$$\|T_\sigma(f,g)\|_Z \leq A\|f\|_X\|g\|_Y$$

for all $f \in \mathcal{S} \cap X$ and all $g \in \mathcal{S} \cap Y$, then we write

$$T_\sigma : X \times Y \to Z.$$

The smallest constant A is denoted by $\|T_\sigma\|_{X\times Y\to Z}$.

If $\mathcal{A}$ is a class of multipliers, we denote by $\mathrm{Op}(\mathcal{A})$ the class of all operators T_σ corresponding to $\sigma \in \mathcal{A}$. If $T_\sigma : X \times Y \to Z$ for all $\sigma \in \mathcal{A}$, then we write

$$\mathrm{Op}(\mathcal{A}) \subset B(X \times Y \to Z).$$

Example 1. Cauchy integral on a curve:

$$\int_{\mathbb{R}} \frac{f(y)}{x-y+i(A(x)-A(y))} dy,$$

where A is a given function on $\mathbb{R}$ with $A' \in L^\infty(\mathbb{R})$.
If $\|A'\|_{L^\infty}$ is small,

$$\int_{\mathbb{R}} \frac{f(y)}{x-y+i(A(x)-A(y))} dy = \sum_{k=0}^{\infty} (-i)^k \int_{\mathbb{R}} \frac{(A(x)-A(y))^k}{(x-y)^{k+1}} f(y) dy.$$

The term for $k=0$ is the Hilbert transform.
The term for $k=1$

$$C_A f(x) = \int_{\mathbb{R}} \frac{A(x)-A(y)}{(x-y)^2} f(y) dy$$

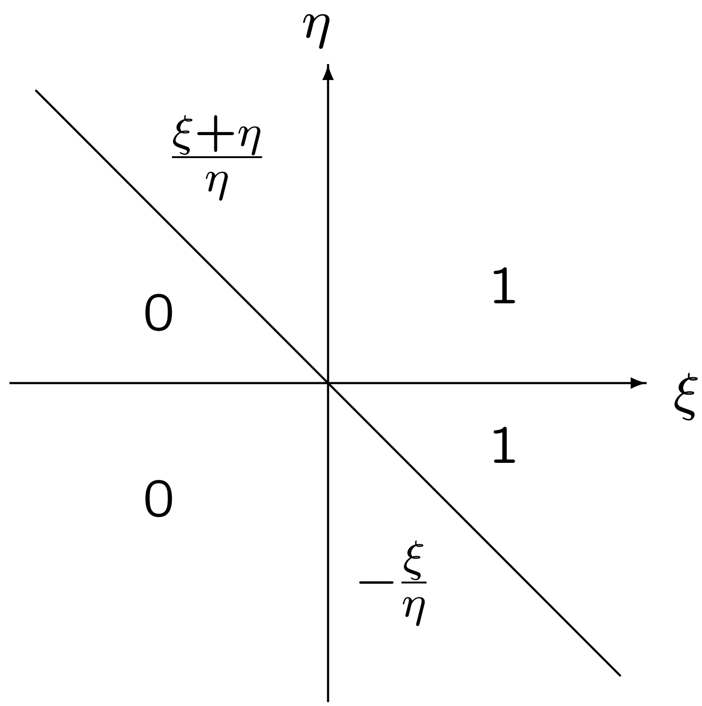
is called **Calderón's commutator**.
If we write $a = A'$, then $A(x) - A(y) = \int_y^x a(t)\, dt$ and

$$C_A f(x) = \int_{\mathbb{R}\times\mathbb{R}} e^{2\pi i x(\xi+\eta)} m(\xi,\eta) \widehat{f}(\xi) \widehat{a}(\eta)\, d\xi d\eta,$$

where

$$m(\xi,\eta) = -\pi i \int_0^1 \operatorname{sign}(\xi + t\eta)\, dt$$

$$= -\pi i \begin{cases} 0 & \text{if} \quad \xi \le 0,\ \xi + \eta \le 0, \\ (\xi + \eta)/\eta & \text{if} \quad \xi \le 0,\ \xi + \eta > 0, \\ -\xi/\eta & \text{if} \quad \xi > 0,\ \xi + \eta \le 0, \\ 1 & \text{if} \quad \xi > 0,\ \xi + \eta > 0. \end{cases}$$



m is homogeneous of degree 0 and Lipschitz continuous in $\mathbb{R}^2 \setminus \{0\}$.

Example 2. For $D^s(f) = (|\xi|^s \widehat{f}\,)^\vee$, the inequality

$$\|D^s(fg)\|_{L^p} \lesssim \|D^s f\|_{L^p} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{L^p}, \quad 1 < p < \infty, \tag{1}$$

is called the **Kato-Ponce inequality**.

To prove this inequality, notice that

$$D^s(fg) = \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{2\pi i x\cdot(\xi+\eta)} |\xi + \eta|^s \, \widehat{f}(\xi) \widehat{g}(\eta) \, d\xi d\eta.$$

Take functions $\Phi_1(\xi, \eta)$ and $\Phi_2(\xi, \eta)$ such that

$$\begin{aligned}
&\Phi_1, \Phi_2 \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}), \\
&\Phi_1 \text{ and } \Phi_2 \text{ are homogeneous of degree 0}, \\
&\xi \neq 0 \text{ on } \operatorname{supp} \Phi_1, \\
&\eta \neq 0 \text{ on } \operatorname{supp} \Phi_2, \\
&\Phi_1 + \Phi_2 = 1 \text{ on } \mathbb{R}^{2n} \setminus \{0\},
\end{aligned}$$

and write

$$
\begin{aligned}
&D^s(fg)\\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{2\pi i x\cdot(\xi+\eta)}\, \Phi_1(\xi,\eta)\frac{|\xi+\eta|^s}{|\xi|^s}\,(D^s f)^\wedge(\xi)\widehat{g}(\eta)\, d\xi d\eta\\
&\quad + \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{2\pi i x\cdot(\xi+\eta)}\, \Phi_2(\xi,\eta)\frac{|\xi+\eta|^s}{|\eta|^s}\,\widehat{g}(\eta)(D^s g)^\wedge(\eta)\, d\xi d\eta.
\end{aligned}
$$

The Fourier multipliers

$$
m_1(\xi,\eta) = \Phi_1(\xi,\eta)\frac{|\xi+\eta|^s}{|\xi|^s}, \quad m_2(\xi,\eta) = \Phi_2(\xi,\eta)\frac{|\xi+\eta|^s}{|\eta|^s},
$$

are not smooth around $\xi+\eta=0$ if s is not an even integer.

The following is the most fundamental result on the boundedness of bilinear Fourier multiplier operators.

Theorem A. (Coifman-Meyer 1978, Kenig-Stein 1999, Grafakos-Kalton 2001) If

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \le C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha|-|\beta|} \tag{A-1}$$

for all α, β, then

$$T_m : H^p \times H^q \to L^r \tag{A-2}$$

for all $p, q, r \in (0, \infty]$ satisfying $1/p + 1/q = 1/r$, where (A-2) **should be replaced by $T_m : L^\infty \times L^\infty \to BMO$ when $p = q = r = \infty$.**

H^p, $p > 0$, are the real Hardy spaces.
Recall that $H^p = L^p$ if $1 < p \le \infty$.

The condition (A-1) is sometimes called the **Hörmander-Mihlin type condition**.

1-2. Nonsmooth multipliers of Hörmander-Mihlin type

We want to refine Theorem A so that we require only limited differentiability of m.

The first results in this direction was given by Tomita in 2010.

We use the function Ψ such that

$$\begin{cases} \Psi \in C_0^\infty(\mathbb{R}^{2n}), \quad \operatorname{supp}\Psi \subset \{\zeta \in \mathbb{R}^{2n} \mid 2^{-1} \le |\zeta| \le 2\}, \\ \displaystyle\sum_{j\in\mathbb{Z}} \Psi(\zeta/2^j) = 1 \ \text{ for all } \ \zeta \in \mathbb{R}^{2n} \setminus \{0\}. \end{cases}$$

For functions on $\mathbb{R}^{2n}$, we define the Sobolev norm by

$$\|G\|_{W^s(\mathbb{R}^{2n})} = \|(1+|z|)^s \widehat{G}(z)\|_{L^2_z(\mathbb{R}^{2n})}.$$

Theorem B. (Tomita 2010) If $s > n$, then

$$\|T_\sigma\|_{L^p \times L^q \to L^r} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot)\Psi(\cdot)\|_{W^s(\mathbb{R}^{2n})}$$

for all $1 < p, q, r < \infty$ satisfying $1/p + 1/q = 1/r$.

Grafakos-Si (2012) extended Tomita's theorem to $r \leq 1$ by using L^λ-based Sobolev norm with $1 < \lambda \leq 2$.

We can refine Theorem B by using the **product type Sobolev norm**

$$\|G\|_{W^{(s_1,s_2)}((\mathbb{R}^n)^2)} = \|(1+|z_1|)^{s_1}(1+|z_2|)^{s_2}\widehat{G}(z_1,z_2)\|_{L^2_{z_1,z_2}((\mathbb{R}^n)^2)}.$$

Theorem C. (M.-Tomita 2013) Let $0 < p, q, r \le \infty$ satisfy $1/p + 1/q = 1/r$. If $s_1, s_2 > n/2$ and

$$s_1 > n/p - n/2, \quad s_2 > n/q - n/2, \quad s_1 + s_2 > n/p + n/q - n/2,$$

then

$$\|T_\sigma\|_{H^p \times H^q \to L^r} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j(\xi,\eta))\Psi(\xi,\eta)\|_{W^{(s_1,s_2)}((\mathbb{R}^n)^2)}, \qquad \text{(C-1)}$$

where we replace $H^p \times H^q \to L^r$ by $L^\infty \times L^\infty \to BMO$ in the case $p = q = r = \infty$. Conversely, if (C-1) **holds with the same replacement in the case $p = q = r = \infty$, then $s_1, s_2 \ge n/2$ and**

$$s_1 \ge n/p - n/2, \quad s_2 \ge n/q - n/2, \quad s_1 + s_2 \ge n/p + n/q - n/2.$$

2. Multipliers of exotic class

2-1. Bilinear Hörmander class

Definition 1. For $m \in \mathbb{R}$ and $0 \le \rho \le 1$, the class $BS_{\rho}^{\langle m \rangle}(\mathbb{R}^n)$ is defined to be the set of all $\sigma(\xi, \eta) \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ that satisfy

$$\left| \partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} \sigma(\xi, \eta) \right| \le C_{\alpha,\beta} (1 + |\xi| + |\eta|)^{m - \rho(|\alpha| + |\beta|)}.$$

This class $BS_{\rho}^{\langle m \rangle}$ (in a generalized form for symbols of bilinear pseudo-differential operators) was introduced and studied by
Bényi-Maldonado-Naibo-Torres (2010),
Bényi-Bernicot-Maldonado-Naibo-Torres (2013),
Michalowski-Rule-Staubach (2014).

In the case $\rho = 1$, the class $BS_1^{\langle 0 \rangle}(\mathbb{R}^n)$ is the class of Hörmander-Mihlin type multipliers, which are considered in Section 1. In this case, the operators in the class $\mathrm{Op}(BS_1^{\langle 0 \rangle})$ have appropriate integral kernel and covered by the **bilinear Calderón-Zygmund theory** given by Grafakos-Torres (2002). The results are parallel to the linear case. In particular, we have the following.

Theorem D. ($\subset$ Theorem A)
$\boldsymbol{\mathrm{Op}(BS_1^{\langle 0 \rangle}) \subset B(H^p \times H^q \to L^r)}$ **for all** $\boldsymbol{0 < p, q, r \le \infty}$ **satisfying** $\boldsymbol{1/p + 1/q = 1/r > 0}$**. Also** $\boldsymbol{\mathrm{Op}(BS_1^{\langle 0 \rangle}) \subset B(L^\infty \times L^\infty \to BMO)}$**.**

However, in the case $0 \le \rho < 1$, the bilinear operators and the linear operators have different features.
The class $BS_\rho^{\langle m \rangle}$ with $0 \le \rho < 1$ is sometimes called the **exotic class**.

Recall the case of linear operators. The linear Fourier multiplier operator $\sigma(D)$,

$$\sigma(D)f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\sigma(\xi)\widehat{f}(\xi)\,d\xi, \qquad x \in \mathbb{R}^n,$$

is bounded $L^2 \to L^2$ if and only if $\sigma \in L^\infty$ (by Plancherel's theorem). Extension of this result to the linear pseudo-differential operators,

$$\sigma(X, D)f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\sigma(x, \xi)\widehat{f}(\xi)\,d\xi, \qquad x \in \mathbb{R}^n,$$

is the following well-known theorem.

Theorem E. (Calderón-Vaillancourt 1972) The linear pseudo -differential operator $\boldsymbol{\sigma(X, D)}$ is bounded $\boldsymbol{L^2 \to L^2}$ if the symbol $\boldsymbol{\sigma}$ satisfies $\boldsymbol{\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \le C_{\alpha,\beta}}$.

Naive generalization of this theorem to the bilinear case fails.

Theorem F. (Bényi-Torres 2004) There exists a function $\sigma = \sigma(\xi, \eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq C_{\alpha,\beta}$$

for all α, β, (i.e., $\sigma \in BS_0^{\langle 0 \rangle}(\mathbb{R}^n)$ in our notation), and that T_σ is not bounded from $L^p \times L^q$ to L^r for any $p, q, r \in [1, \infty)$ satisfying $1/p + 1/q = 1/r$.

This theorem says that the relation

$$\mathrm{Op}(BS_0^{\langle m \rangle}(\mathbb{R}^n)) \subset B(L^p \times L^q \to L^r)$$

holds only if $m < 0$.

2-2. Critial m for the exotic class

Definition 2. Let $0 \le \rho < 1$, $0 < p, q \le \infty$, and $1/p + 1/q = 1/r$. We define

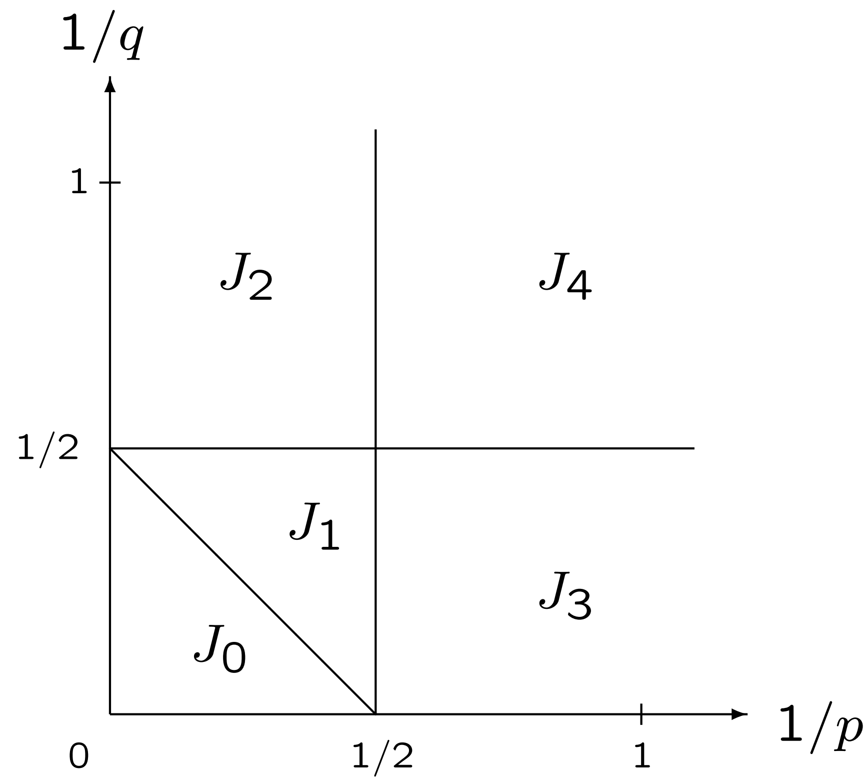
$$m_\rho(p,q) = \sup\{m \in \mathbb{R} : \mathrm{Op}(BS_\rho^{\langle m \rangle}(\mathbb{R}^n)) \subset B(H^p \times H^q \to L^r)\},$$

where $H^p \times H^q \to L^r$ should be replaced by $L^\infty \times L^\infty \to BMO$ in the case $p = q = r = \infty$.

Theorem G. For $0 \le \rho < 1$, $0 < p, q \le \infty$, and $1/p + 1/q = 1/r$. Then $m_\rho(p,q) = (1-\rho) m_0(p,q)$ and

$$\boldsymbol{m_0(p,q) = -n \max\left\{\frac{1}{2}, \frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{r}, \frac{1}{r} - \frac{1}{2}\right\}.}$$

Notice that $m_0(p,q) \le -n/2$ for all (p,q).



$$m_0(p,q) = \begin{cases} n/r - n & \text{if} \quad (1/p, 1/q) \in J_0, \\ -n/2 & \text{if} \quad (1/p, 1/q) \in J_1, \\ -n/q & \text{if} \quad (1/p, 1/q) \in J_2, \\ -n/p & \text{if} \quad (1/p, 1/q) \in J_3, \\ n/2 - n/r & \text{if} \quad (1/p, 1/q) \in J_4. \end{cases}$$

The critical order $m_\rho(p,q)$ were implicitly given in the works of several authors. In particular, Boundedness for the case $m < m_0(p,q)$ were given by
Michalowski-Rule-Staubach (2014): $\mathrm{Op}(BS_0^{\langle m \rangle}) \subset B(L^2 \times L^2 \to L^1)$ for $m < -n/2 = m_0(2,2)$,
and by
Bényi-Bernicot-Maldonado-Naibo-Torres (2013): $\mathrm{Op}(BS_0^{\langle m \rangle}) \subset B(L^p \times L^q \to L^r)$ for $1 \le p,q,r \le \infty$, $1/p + 1/q = 1/r$, and $m < m_0(p,q)$.

Problem: how about the critical case $m = m_\rho(p,q)$?

Theorem H. (M.-Tomita 2013, Naibo 2015, M.-Tomita 2017, 2018) Let $0 \le \rho < 1$, $0 < p,q,r \le \infty$, $1/p + 1/q = 1/r$, and $m = m_\rho(p,q)$. Then

$$\mathrm{Op}(BS_\rho^{\langle m \rangle}(\mathbb{R}^n)) \subset B(H^p \times H^q \to L^r),$$

where $H^p \times H^q \to L^r$ should be replaced by $L^\infty \times L^\infty \to BMO$ in the case $p = q = r = \infty$.

3. Generalization of the bilinear exotic class

In this section, we shall mainly consider the boundedness of bilinear Fourier multiplier operators on $L^2 \times L^2$, which is the most fundamental estimate.

3-1. The class $BS_{\mathbf{0}}^{W}(\mathbb{R}^n)$ with general W

Recall the following theorem, which is a part of Theorem H.

Theorem I. (M.-Tomita 2013) If $\sigma \in BS_{\mathbf{0}}^{\langle -n/2 \rangle}(\mathbb{R}^n)$, i.e., if

$$|\partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} \sigma(\xi, \eta)| \le C_{\alpha,\beta}(1 + |\xi| + |\eta|)^{-n/2},$$

then $T_{\sigma} : L^2 \times L^2 \to L^1$. The exponent $n/2$ cannot be replaced by a smaller number.

We shall show that the class $BS_0^{\langle -n/2\rangle}(\mathbb{R}^n)$ can be replaced by a wider general class.

Definition 3. Let W be a nonnegative bounded function on $\mathbb{R}^n \times \mathbb{R}^n$. We define $BS_0^W(\mathbb{R}^n)$ to be the set of all C^∞ functions $\sigma = \sigma(\xi, \eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)\right| \le C_{\alpha,\beta} W(\xi, \eta)$$

for all multi-indices α, β. We call W the **weight function**.

Thus

$$BS_0^{\langle -n/2\rangle}(\mathbb{R}^n) = BS_0^W(\mathbb{R}^n) \quad \text{with} \quad W(\xi, \eta) = (1 + |\xi| + |\eta|)^{-n/2}.$$

Definition 4. $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ denotes the set of all nonnegative functions V on $\mathbb{Z}^n \times \mathbb{Z}^n$ for which there exists a constant $c \in (0, \infty)$ such that the inequality

$$\sum_{\mu,\nu \in \mathbb{Z}^n} V(\mu,\nu) A(\mu+\nu) B(\mu) C(\nu) \\ \leq c \|A\|_{\ell^2(\mathbb{Z}^n)} \|B\|_{\ell^2(\mathbb{Z}^n)} \|C\|_{\ell^2(\mathbb{Z}^n)}$$

holds for all nonnegative functions A, B, C on $\mathbb{Z}^n$.

Definition 5. For a nonnegative bounded function V on $\mathbb{Z}^n \times \mathbb{Z}^n$, we define the function $\widetilde{V}$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\widetilde{V}(\xi,\eta) = \sum_{\mu,\nu \in \mathbb{Z}^n} V(\mu,\nu) \mathbf{1_Q}(\xi-\mu) \mathbf{1_Q}(\eta-\nu),$$

where $Q = (-1/2, 1/2]^n$.

Theorem J. **(Kato-M.-Tomita 2019) Let V and $\widetilde{V}$ be as above.**
(1) If there exists an $r \in (0, \infty)$ such that

$$\mathrm{Op}(BS_0^{\widetilde{V}}(\mathbb{R}^n)) \subset B(L^2 \times L^2 \to L^r), \tag{2}$$

then $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$.
(2) Conversely, if $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$, then

$$\mathrm{Op}(BS_0^{\widetilde{V}}(\mathbb{R}^n)) \subset B(L^2 \times L^2 \to L^r) \ \ \textbf{for} \ \ 1 \le r \le 2.$$

Thus, in particular,

$$\begin{aligned}
&V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n) \\
&\Leftrightarrow \mathrm{Op}(BS_0^{\widetilde{V}}(\mathbb{R}^n)) \subset B(L^2 \times L^2 \to L^1) \\
&\Leftrightarrow \mathrm{Op}(BS_0^{\widetilde{V}}(\mathbb{R}^n)) \subset B(L^2 \times L^2 \to L^r) \quad \textbf{for} \quad 1 \le r \le 2.
\end{aligned}$$

Theorem K. (Kato-M.-Tomita 2019)

(1) All nonnegative functions in the Lorentz class $\ell^{4,\infty}((\mathbb{Z}^n)^2)$ belong to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$. In particular the function

$$V(\mu,\nu) = (1+|\mu|+|\nu|)^{-n/2}$$

belongs to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$.

(2) If $V_1, V_2 \in \ell^{4,\infty}(\mathbb{Z}^n)$ are nonnegative, then the function $V_1(\mu)V_2(\nu)$ belongs to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$. In particular the function

$$V(\mu,\nu) = (1+|\mu|)^{-n/4}(1+|\nu|)^{-n/4}$$

belongs to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$.

(3) If $V_{1,j}, V_{2,j} \in \ell^{4,\infty}(\mathbb{Z})$ are nonnegative, $j = 1, \ldots, n$, then the function $\prod_{j=1}^{n} V_{1,j}(\mu_j)V_{2,j}(\nu_j)$belongs to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$. In particular the function

$$V(\mu,\nu) = \prod_{j=1}^{n} (1+|\mu_j|)^{-1/4}(1+|\nu_j|)^{-1/4}$$

belongs to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$.

The following corollary follows from Theorem J with the weight function $V(\mu,\nu) = (1+|\mu|)^{-n/4}(1+|\nu|)^{-n/4}$.

Corollary 1. $\mathrm{Op}(BS_0^{\langle 0 \rangle}(\mathbb{R}^n)) \subset B(W^{n/4} \times W^{n/4} \to L^1 \cap L^2)$**, where** $W^{n/4} = W^{n/4}(\mathbb{R}^n)$ **denotes the Sobolev space,**

$$\|f\|_{W^{n/4}(\mathbb{R}^n)} = \|(1+|\xi|)^{n/4}\widehat{f}(\xi)\|_{L^2(\mathbb{R}^n)}.$$

The next theorem can be proved by the use of Theorems J and K.

Theorem L. (Grafakos-He-Slavíková 2018, Kato-M.-Tomita 2019) If $|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta}$ for all α, β and if $\sigma \in L^q(\mathbb{R}^{2n})$ for some $q < 4$, then the operator T_σ is bounded from $L^2 \times L^2$ to $L^1 \cap L^2$.

It is known that we cannot take $q = 4$ in the above theorem.

Theorem M. (Slavíková 2019) The assertion of Theorem L does not hold for $q = 4$, i.e., there exists a $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta) \in L^\infty(\mathbb{R}^{2n})$ for all α, β and $\sigma \in L^4(\mathbb{R}^{2n})$ and the corresponding bilinear Fourier multiplier operator T_σ is not bounded from $L^2 \times L^2$ to L^1.

3-2. The amalgam space

One of the ideas to prove Theorem J was to use the amalgam norm, which is defined as follows.

Definition 6. For $0 < q \le \infty$ and for measurable functions f on $\mathbb{R}^n$, the **amalgam norm** is defined by

$$\|f\|_{(L^2,\ell^q)} = \left(\sum_{\nu \in \mathbb{Z}^n} \|f(x+\nu)\|^q_{L^2_x(Q)} \right)^{1/q}$$

with the usual modification in the case $q = \infty$, where $Q = (-1/2, 1/2]^n$. The class of all f with $\|f\|_{(L^2,\ell^q)} < \infty$ is defined to be the amalgam space $(L^2, \ell^q) = (L^2, \ell^q)(\mathbb{R}^n)$.

Proposition 1.

$$\boldsymbol{(L^2, \ell^q) \supset L^q \quad \textbf{if} \quad 2 \le q \le \infty,}$$

$$\boldsymbol{(L^2, \ell^q) \subset L^q \quad \textbf{if} \quad 0 < q \le 2.}$$

Theorem N. (Kato-M.-Tomita 2019)
(1) If there exist $0 < p, q, r \le \infty$ such that

$$\mathrm{Op}(BS_0^{\widetilde{V}}(\mathbb{R}^n)) \subset B\big((L^2, \ell^p) \times (L^2, \ell^q) \to (L^2, \ell^r)\big), \qquad \text{(N-1)}$$

then $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$.
(2) Conversely, if $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$, then (N-1) holds for all $0 < p, q, r \le \infty$ satisfying $1/p + 1/q \ge 1/r$.

Summarizing Theorems J and N, we have

$$\begin{aligned}
&V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n) \\
&\Leftrightarrow \mathrm{Op}(BS_0^{\widetilde{V}}(\mathbb{R}^n)) \subset B(L^2 \times L^2 \to L^1) \\
&\Leftrightarrow \mathrm{Op}(BS_0^{\widetilde{V}}(\mathbb{R}^n)) \subset B(L^2 \times L^2 \to L^r) \quad \text{for} \quad 1 \le r \le 2. \\
&\Leftrightarrow \mathrm{Op}(BS_0^{\widetilde{V}}(\mathbb{R}^n)) \subset B\big((L^2, \ell^p) \times (L^2, \ell^q) \to (L^2, \ell^r)\big) \\
&\qquad\qquad \text{for} \quad 0 < p, q, r \le \infty, \;\; 1/p + 1/q \ge 1/r.
\end{aligned}$$

3-3. Proof of Theorem J (1)

Let V be a nonnegative bounded function on $\mathbb{Z}^n \times \mathbb{Z}^n$ and $0 < r < \infty$. We assume $\mathrm{Op}(BS_0^{\widetilde{V}}) \subset B(L^2 \times L^2 \to L^r)$. We shall prove $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$, i.e.,

$$\sum_{\mu,\nu \in \mathbb{Z}^n} V(\mu,\nu) A(\mu+\nu) B(\mu) C(\nu)$$
$$\leq c \|A\|_{\ell^2(\mathbb{Z}^n)} \|B\|_{\ell^2(\mathbb{Z}^n)} \|C\|_{\ell^2(\mathbb{Z}^n)}$$

for all nonnegative functions $A, B, C \in \ell^2(\mathbb{Z}^n)$.

Notice that, by the closed graph theorem, our assumption implies that there exist a positive integer M and a positive constant c such that

$$\|T_\sigma\|_{L^2 \times L^2 \to L^r} \leq c \max_{|\alpha|,|\beta| \leq M} \left\| \widetilde{V}(\xi,\eta)^{-1} \partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta) \right\|_{L^\infty}$$

for all bounded smooth functions σ on $(\mathbb{R}^n)^2$.

Take $\varphi, \widetilde{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \widetilde{\varphi} \subset [-1/2, 1/2]^n, \quad \widetilde{\varphi} = 1 \text{ on } [-1/4, 1/4]^n,$$
$$\operatorname{supp} \varphi \subset [-1/4, 1/4]^n, \quad |\mathcal{F}^{-1}\varphi| \geq 1 \text{ on } [-1/2, 1/2]^n.$$

Let $\{\epsilon_k\}$ be a sequence such that $\epsilon_k = \pm 1$. Consider the multiplier

$$m(\xi, \eta) = \sum_{\mu, \nu \in \mathbb{Z}^n} \epsilon_{\mu+\nu} V(\mu, \nu) \widetilde{\varphi}(\xi - \mu) \widetilde{\varphi}(\eta - \nu).$$

Then

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \leq C_{\alpha,\beta} \widetilde{V}(\xi, \eta)$$

with $C_{\alpha,\beta}$ independent of the sequence $\{\epsilon_k\}$.

We define $f, g \in \mathcal{S}(\mathbb{R}^n)$ by

$$\widehat{f}(\xi) = \sum_{\mu \in \mathbb{Z}^n} B(\mu) \varphi(\xi - \mu),$$
$$\widehat{g}(\eta) = \sum_{\nu \in \mathbb{Z}^n} C(\nu) \varphi(\eta - \nu).$$

Then $\|f\|_{L^2} \approx \|B\|_{\ell^2}$ and $\|g\|_{L^2} \approx \|C\|_{\ell^2}$.

From the situation of the supports of φ and $\widetilde{\varphi}$, we have

$$\begin{aligned} T_m(f,g)(x) &= \sum_{\mu,\nu\in\mathbb{Z}^n} \epsilon_{\mu+\nu} V(\mu,\nu)B(\mu)C(\nu)e^{2\pi i(\mu+\nu)\cdot x}\mathcal{F}^{-1}\varphi(x)^2 \\ &= \sum_k \epsilon_k d_k e^{2\pi i k\cdot x}\mathcal{F}^{-1}\varphi(x)^2 \end{aligned}$$

with

$$d_k = \sum_{\mu+\nu=k} V(\mu,\nu)B(\mu)C(\nu).$$

Our assumption implies

$$\|T_m(f,g)\|_{L^r} \lesssim \|f\|_{L^2}\|g\|_{L^2} \approx \|B\|_{\ell^2}\|C\|_{\ell^2}.$$

We have

$$\|T_m(f,g)\|_{L^r}^r \geq \int_{[-1/2,1/2]^n} \left| \sum_k \epsilon_k d_k e^{2\pi i k\cdot x} \right|^r dx.$$

Hence

$$\int_{[-1/2,1/2]^n} \left| \sum_k \epsilon_k d_k e^{2\pi i k\cdot x} \right|^r dx \lesssim \left(\|B\|_{\ell^2}\|C\|_{\ell^2} \right)^r. \tag{3}$$

Notice that the implicit constant in (3) does not depend on $\{\epsilon_k\}$.

We take the average over all choices of $\epsilon_k = \pm 1$. Then Khintchine's inequality yields

$$\begin{aligned}\left(\sum_k |d_k|^2\right)^{r/2} &= \left\| \sum_{\mu+\nu=k} V(\mu,\nu)B(\mu)C(\nu)\right\|_{\ell^2_k}^r \\ &\lesssim \left(\|B\|_{\ell^2}\|C\|_{\ell^2}\right)^r,\end{aligned}$$

which is equivalent to the desired inequality.

3-4. Proof of Theorem J (2)

We assume $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ and define $\widetilde{V}$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\widetilde{V}(\xi, \eta) = \sum_{\mu, \nu \in \mathbb{Z}^n} V(\mu, \nu) \mathbf{1}_{\mathbf{Q}}(\xi - \mu) \mathbf{1}_{\mathbf{Q}}(\eta - \nu),$$

where $Q = (-1/2, 1/2]^n$. We assume $\sigma \in BS_0^{\widetilde{V}}(\mathbb{R}^n)$ and prove $T_\sigma : L^2 \times L^2 \to (L^2, \ell^1)$.

It can be shown that there exists a function W on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$W(\xi)(1+|\eta|)^{-M} \lesssim W(\xi+\eta) \lesssim W(\xi)(1+|\eta|)^M$$

for some $M \in (0, \infty)$ and

$$W|_{\mathbb{Z}^n \times \mathbb{Z}^n} \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n), \quad \widetilde{V}(\xi) \le W(\xi).$$

We shall use this function W instead of $\widetilde{V}$.
Our assumption implies that $\sigma \in BS_0^W(\mathbb{R}^n)$.

We take the usual Littlewood-Paley functions $\{\psi_k\}_{k\in\mathbb{N}\cup\{0\}}$, which are defined as follows. Take a function ψ such that

$$\begin{cases} \psi \in C_0^\infty(\mathbb{R}^n), \quad \operatorname{supp}\psi \subset \{\xi \in \mathbb{R}^n \mid 2^{-1} \le |\xi| \le 2\}, \\ \displaystyle\sum_{j\in\mathbb{Z}} \psi(\xi/2^j) = 1 \ \text{ for all } \ \xi \in \mathbb{R}^n \setminus \{0\}, \end{cases}$$

and define

$$\psi_k(\xi) = \psi(\xi/2^k), \quad k = 1, 2, \dots,$$
$$\psi_0(\xi) = 1 - \sum_{k=1}^{\infty} \psi_k(\xi).$$

We decompose σ as

$$\begin{aligned}\sigma(\xi,\eta) &= \sum_{k_1,k_2\in\mathbb{N}\cup\{0\}} \psi_{k_1}\Big(D_\xi\Big)\psi_{k_2}\Big(D_\eta\Big)\sigma(\xi,\eta)\\ &= \sum_{k_1,k_2\in\mathbb{N}\cup\{0\}} \sigma_{k_1,k_2}(\xi,\eta),\end{aligned}$$

where

$$\begin{aligned}&\sigma_{k_1,k_2}(\xi,\eta) = \psi_{k_1}\Big(D_\xi\Big)\psi_{k_2}\Big(D_\eta\Big)\sigma(\xi,\eta)\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{2\pi i(\xi\cdot z_1+\eta\cdot z_2)}\psi_{k_1}(z_1)\psi_{k_2}(z_2)\widehat{\sigma}(z_1,z_2)\,dz_1dz_2.\end{aligned}$$

Then

$$\operatorname{supp}\Big(\sigma_{k_1,k_2}\Big)^\wedge \subset \{(z_1,z_2)\in(\mathbb{R}^n)^2 \mid |z_1|\le 2^{k_1+1},\ |z_2|\le 2^{k_2+1}\}$$

and, by virtue of the moderate behavior of W,

$$\Big\|W(\xi,\eta)^{-1}\sigma_{k_1,k_2}(x,\xi,\eta)\Big\|_{L^\infty((\mathbb{R}^n)^2)} \lesssim 2^{-(k_1+k_2)N},$$

where $N>0$ can be taken arbitrarily large.

The following proposition is the essential part of the proof.

Proposition 2. Suppose τ is a bounded continuous function on $(\mathbb{R}^n)^2$ such that

$$\operatorname{supp}\widehat{\tau} \subset \{(z_1, z_2) \in (\mathbb{R}^n)^2 \mid |z_1| \leq 2^{k_1},\ |z_2| \leq 2^{k_2}\}$$

with $k_1, k_2 \in \mathbb{N} \cup \{0\}$. Then

$$\|T_\tau\|_{L^2 \times L^2 \to (L^2, \ell^1)} \lesssim 2^{(k_1+k_n)n/2} \|W(\xi, \eta)^{-1}\tau(\xi, \eta)\|_{L^\infty((\mathbb{R}^n)^2)}.$$

Applying this proposition to $\tau = \sigma_{k_1,k_2}$ and taking sum over $k_1, k_2 \in \mathbb{N} \cup \{0\}$, we conclude that $\|T_\sigma\|_{L^2 \times L^2 \to (L^2, \ell^1)} < \infty$.

In the proof of Proposition 2, we used some ideas of Boulkhemair 1995, who considered sharp L^2 estimates for linear pseudo-differential operators.

3-5. Proof of $\ell^{4,\infty}(\mathbb{Z}^{2n}) \subset \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$

By appropriately extending functions on $\mathbb{Z}^n$ and $\mathbb{Z}^n \times \mathbb{Z}^n$ to functions on $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$, it is sufficient to prove the continuous version

$$\begin{aligned}&\int_{\mathbb{R}^n \times \mathbb{R}^n} V(x,y)A(x+y)B(x)C(y)\,dxdy \\ &\lesssim \|V\|_{L^{4,\infty}(\mathbb{R}^n \times \mathbb{R}^n)}\|A\|_{L^2(\mathbb{R}^n)}\|B\|_{L^2(\mathbb{R}^n)}\|C\|_{L^2(\mathbb{R}^n)}\end{aligned} \tag{4}$$

for nonnegative functions $V, A, B, C \in L^2(\mathbb{R}^n)$.

It is known that the inequality

$$\begin{aligned}&\int_{\mathbb{R}^n \times \mathbb{R}^n} V(x,y)A(x+y)B(x)C(y)\,dxdy \\ &\lesssim \|V\|_{L^{q_0}(\mathbb{R}^n \times \mathbb{R}^n)}\|A\|_{L^{q_1}(\mathbb{R}^n)}\|B\|_{L^{q_2}(\mathbb{R}^n)}\|C\|_{L^{q_3}(\mathbb{R}^n)}\end{aligned}$$

holds if and only if

$$\begin{aligned}&\frac{2}{q_0}+\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=2,\\ &0 \le \frac{1}{q_i} \le 1-\frac{1}{q_0} \le 1, \quad i=1,2,3.\end{aligned}$$

By the real interpolation for multilinear operators (S. Janson), it follows that the inequality

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n} V(x,y)A(x+y)B(x)C(y)\,dxdy\\&\lesssim \|V\|_{L^{q_0,r_0}(\mathbb{R}^{2n})}\|A\|_{L^{q_1,r_1}(\mathbb{R}^n)}\|B\|_{L^{q_2,r_2}(\mathbb{R}^n)}\|C\|_{L^{q_3,r_3}(\mathbb{R}^n)}\end{aligned} \tag{5}$$

holds if

$$\begin{aligned}&\frac{2}{q_0}+\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=2,\\&0<\frac{1}{q_i}<1-\frac{1}{q_0}<1,\quad i=1,2,3,\\&\frac{1}{r_0}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=1.\end{aligned}$$

In particular, by taking $q_0=4$, $q_1=q_2=q_3=2$, $r_0=r_1=\infty$, and $r_2=r_3=2$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n} V(x,y)A(x+y)B(x)C(y)\,dxdy\\&\lesssim \|V\|_{L^{4,\infty}(\mathbb{R}^n\times\mathbb{R}^n)}\|A\|_{L^{2,\infty}(\mathbb{R}^n)}\|B\|_{L^2(\mathbb{R}^n)}\|C\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

which *a fortiori* implies the desired inequality (4).

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