On pseudoconformal blow-up solutions to the self-dual Chern-Simons-Schrödinger equation

Soonsik Kwon

Korea Advanced Institute of Science and Technology

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joint with Kihyun Kim(KAIST)

Outline

Chern-Simons-Schrödinger Equation

Pseudoconformal Blow-up Solutions

Strategy of the proof

Chern-Simons-Schrödinger equation

We consider the **Chern-Simons-Schrödinger equation**:

$$\begin{cases} \mathbf{D}_t\phi = i\mathbf{D}_j\mathbf{D}_j\phi + ig|\phi|^2\phi, \\ F_{01} = -\mathrm{Im}(\overline{\phi}\mathbf{D}_2\phi), \\ F_{02} = \mathrm{Im}(\overline{\phi}\mathbf{D}_1\phi), \\ F_{12} = -\frac{1}{2}|\phi|^2, \end{cases}$$

with a scalar field $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$, covariant derivatives $\mathbf{D}_\alpha = \partial_\alpha + iA_\alpha$ for $\alpha \in \{0,1,2\}$, (real-valued) connection 1-form $A = A_0dt + A_1dx_1 + A_2dx_2$, and curvature 2-form $F_{jk} := \partial_j A_k - \partial_k A_j$.

- Non-relativistic Lagrangian theory,
- Planar physical phenomena, e.g. quantum Hall effect and high temperature superconductivity,
- **Gauge invariance**: for any $\chi : \mathbb{R}^{1+2} \to \mathbb{R}$,

$$(\phi, A) \mapsto (e^{i\chi}\phi, A - d\chi)$$

- See Jackiw-Pi ('90 PRL, '90 PRD)

Coulomb gauge and equivariance condition

Coulomb gauge condition

$$\partial_1 A_1 + \partial_2 A_2 = 0 \quad \text{or} \quad A_r = 0.$$

Equivariance ansatz:

$$\phi(t,x) = e^{im\theta} u(t,r).$$

- $m \in \mathbb{Z}$ is called the **equivariance index**.
- The full equation becomes

$$i\partial_t u + \left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{m^2}{r^2}\right) u - \frac{2mA_\theta}{r} u - A_\theta^2 u - A_0 u + g|u|^2 u = 0$$

with connection components

$$\begin{cases} A_\theta = -\dfrac{1}{2}\displaystyle\int_0^r |u|^2 r' dr', \\ A_0 = -\displaystyle\int_r^\infty (m + A_\theta)|u|^2 \frac{dr'}{r'}. \end{cases}$$

Bogomol'nyi operator

Bogomol'nyi operator

$$\mathbf{D}_+ u := \mathbf{D}_+^{(u)} u = \left(\partial_r - \frac{m + A_\theta[u]}{r}\right) u.$$

- It is the radial part of $\mathbf{D}_1 + i\mathbf{D}_2$.
- **Hamiltonian structure**:

  $$i\partial_t \phi = \frac{\delta E}{\delta \phi},$$

  where $\frac{\delta E}{\delta \phi}$ is the Fréchet derivative computed under the real inner product $(u, v)_r := \mathrm{Re} \int_{\mathbb{R}^2} \overline{u} v$.
- **Energy functional** has the expression

  $$E[u] = \frac{1}{2} \int |\mathbf{D}_+ u|^2 + \frac{1-g}{4} \int |u|^4.$$

  The factor **1** arises from the curvature term $F_{r\theta}$. Thus, $g < 1$ is defocusing and $g \geq 1$ is focusing.
- **GWP** and **Scattering** under equivariance: **Liu-Smith** ('16)
  - $g < 1$: all L^2-data
  - $g \geq 1$: L^2-data whose charge is less than that of the ground state.
- The borderline case $g = 1$ is called the **self-dual** case.

Equivariant self-dual CSS

From now on, we restrict to the self-dual case $g = 1$.

- **(CSS)** in various forms:

$$i\partial_t u = -\Big(\partial_{rr} + \frac{1}{r}\partial_r\Big)u + \Big(\frac{m + A_\theta}{r}\Big)^2 u + A_0 u - |u|^2 u, \quad \text{(CSS)}$$

$$i\partial_t u + \Delta_m u = -|u|^2 u + \frac{2mA_\theta}{r^2}u + \frac{A_\theta^2}{r^2}u + A_0 u, \quad \text{(lin./nonlin.)}$$

$$i\partial_t u = L_u^* \mathbf{D}_+^{(u)} u. \quad \text{(self-dual)}$$

- $\Delta_m := \partial_{rr} + \frac{1}{r}\partial_r - \frac{m^2}{r^2}$ is the Laplacian for m-equivariant functions.
- L_u is the linearized operator of $\mathbf{D}_+^{(u)} u = \partial_r - \frac{1}{r}(m + A_\theta[u])$.
  $L_u^* f = D_+^{(u)*} f + u \int_r^\infty \mathrm{Re}\,\overline{u} f \, dr'$ is its adjoint.
- Connection components:

$$\begin{cases} A_\theta[u] = -\dfrac{1}{2}\displaystyle\int_0^r |u|^2 r' dr', \\ A_0[u] = -\displaystyle\int_r^\infty (m + A_\theta[u])|u|^2 \frac{dr'}{r'}. \end{cases}$$

- **Tail** of A_θ:

$$A_\theta(0) = 0, \qquad A_\theta(r) \downarrow A_\theta(\infty) = -\frac{1}{4\pi} M[u] \neq 0.$$

Symmetries and conservation laws

- Symmetries:

$$u(t,r) \mapsto \begin{cases} e^{i\theta} u(t,r) & \text{(phase rotation)} \\ \lambda u(\lambda^2 t, \lambda r) & \text{(L^2-critical scaling)} \\ \frac{1}{t} e^{i\frac{|x|^2}{4t}} \phi(-\frac{1}{t}, \frac{x}{t}) & \text{(pseudoconformal)} \end{cases}$$

  Also, there are space/time translation, spatial rotation, time reversal, and Galilean boost.

- Charge and Energy:

$$M[u] = \int |u|^2, \qquad E[u] = \begin{cases} \int \frac{1}{2}|\partial_r u|^2 + \frac{1}{2}\Big(\frac{m + A_\theta}{r}\Big)^2 |u|^2 - \frac{1}{4}|u|^4, \\ \int \frac{1}{2}|\mathbf{D}_+ u|^2. \end{cases}$$

- Virial Identities:

$$\begin{cases} \partial_t \Big(\int |r|^2 |u|^2 \Big) = 4 \int_{\mathbb{R}^2} \mathrm{Im}(\overline{u} \cdot r \partial_r u), \\ \partial_t \Big(\int_{\mathbb{R}^2} \mathrm{Im}(\overline{u} \cdot r\partial_r u) \Big) = 4E. \end{cases}$$

Cauchy theory

The evolution by (CSS) should be understood **modulo gauge equivalence**. To study the Cauchy theory of (CSS), we should fix one representative (ϕ, A) from its (gauge-)equivalence class.

- Under the **Coulomb gauge**:
  Large data H^1-subcritical LWP (Berge-de Bouard-Saut '95, Huh '13, Lim '18)
  Sufficient conditions for blow-up (Berge-de Bouard-Saut '95)
  Explicit blow-up solutions for $g = 1$ (Jackiw-Pi '90, Huh '09)
  Decay estimates for small data (Oh-Pusateri '15)
- **Equivariance** under the Coulomb gauge:
  Large data L^2-critical GWP and Scattering (Liu-Smith '16)
- Under the **Heat gauge**:
  Small data H^ε-subcritical LWP for any $\varepsilon > 0$ (Liu-Smith-Tataru '14)

Static solution

A solution $u(t,r)$ to (CSS) is said to be **static** if u is independent of t. From

$$i\partial_t u = \frac{\delta E}{\delta u} \qquad \text{and} \qquad E[u] = \frac{1}{2}\int |\mathbf{D}_+ u|^2 \geq 0.$$

- **FACT: A solution is static if and only if of zero energy.**
- **It satisfies the Bogomol'nyi equation**;

$$\mathbf{D}_+ u = \left(\partial_r - \frac{m + A_\theta[u]}{r}\right) u = 0, \qquad A_\theta[u] = -\frac{1}{2}\int_0^r |u|^2 r' dr'.$$

  This is a **nonlocal first-order** ODE.
- **Explicit** m**-equivariant static solutions** (unique up to phase/scaling):

$$Q(r) = \sqrt{8}(m+1)\frac{r^m}{1 + r^{2(m+1)}}.$$

  Note Q has degeneracy r^m at 0 and polynomially decays $r^{-(m+2)}$ at ∞.

$$A_\theta(Q)(\infty) = -2(m+1) = -\frac{1}{4\pi}M(Q).$$

- Applying the pseudoconformal transformation to the static solution Q, we have an **explicit finite-time blow-up** solution

$$S(t,r) := \frac{1}{|t|}Q\left(\frac{r}{|t|}\right)e^{-i\frac{r^2}{4|t|}}, \qquad \forall t < 0.$$

  And the blow-up rate is

$$\|\nabla S(t)\|_{L^2} \sim \frac{1}{|t|}.$$

  We call this blow-up rate the **pseudoconformal blow-up rate**.

Outline

Chern-Simons-Schrödinger Equation

Pseudoconformal Blow-up Solutions

Strategy of the proof

Main Results

Question: *How generic is the pseudoconformal blow-up solutions?*
Let $m \geq 1$. Let z^* be a prescribed asymptotic profile satisfying **(H)** with $0 < \alpha^* \ll 1$.

Assumption (H)

$-(m+2)$-equivariant function $\widetilde{z}^* := e^{-i(2m+2)\theta} z^*$ satisfies $\|\widetilde{z}^*\|_{H^k_{-(m+2)}} < \alpha^*$ for some $k = k(m) > m+3$.

- H^k_m is the usual Sobolev space H^k restricted on m-equivariant functions.
- Roughly speaking, z^* is smooth, small, and degenerate at the origin $|z^*(r)| \lesssim \alpha^* r^{m+2}$.

Our main results are threefold:

1. (**Existence**) There exists a pseudoconformal blow-up solution u with the asymptotic profile z^*.
2. (**Uniqueness**) Such a solution u is unique in a suitable class;
3. (**Instability**) Such a solution u shows a rotational instability.

Existence

Theorem (Kim and K. '19)

*Let $m \geq 1$. Let z^* be an m-equivariant profile satisfying **(H)** with sufficiently **small** $\alpha^* > 0$. Then, there exists a solution u to (CSS) on $(-\infty, 0)$ such that*

$$\Big[u(t,r) - \frac{1}{|t|} Q\Big(\frac{r}{|t|}\Big) e^{-i\frac{r^2}{4|t|}}\Big] e^{im\theta} \to z^* \quad \text{in } H_m^1 \text{ as } t \to 0^-.$$

Moreover, u scatters backward in time. Indeed, u satisfies

$$\|u(t,r) - \frac{1}{|t|} Q_{|t|}\Big(\frac{r}{|t|}\Big) e^{i\gamma_{\mathrm{cor}}(t)} - z(t,r)\|_{H_m^1} \lesssim \alpha^* |t|^m.$$

- Here, $z(t,r)$ is a solution to (zCSS) with the initial data $z(0,r) = z^*(r)$. More precisely, an $-(m+2)$-equivariant function $\tilde{z}(t,x) := e^{-i(2m+2)\theta} z(t,x)$ solves $-(m+2)$-equivariant (CSS) with the initial data $\tilde{z}(0,x) = e^{-i(2m+2)\theta} z^*(x)$.
- $\gamma_{\mathrm{cor}}(t)$ is a phase correction term, whose explicit formula is described in terms of z.

Uniqueness

Theorem (Kim and K. '19)

Let m and z^* be as above. Assume two H^1_m-solutions u_1 and u_2 to (CSS) satisfy

$$\|u_j(t,r) - \frac{1}{|t|}Q_{|t|}\left(\frac{r}{|t|}\right)e^{i\gamma_{\mathrm{cor}}(t)} - z(t,r)\|_{H^1_m} \leq c|t|$$

for all $j = 1, 2$ and t near zero, for sufficiently small $\alpha^* > 0$ and $c > 0$. Then, $u_1 = u_2$.

- In particular, if $0 < \alpha^* \ll c \ll 1$, then the solution constructed in the above is unique.

Instability

Theorem (Kim and K. '19)

Let m and z^* be as above. Let u be the pseudoconformal blow-up solution constructed in the above. There exists $\eta^* > 0$ and one-parameter family of H^1_m-solutions $\{u^{(\eta)}\}_{\eta\in[0,\eta^*]}$ to (CSS) with the following properties.

- $u^{(0)} = u$,
- *For $\eta > 0$, $u^{(\eta)}$ scatters both forward and backward in time,*
- *The map $\eta \in [0, \eta^*] \mapsto u^{(\eta)}$ is continuous in the $C_{(-\infty,0),\mathrm{loc}} H^{1-}$ topology,*
- *The family $\{u^{(\eta)}\}_{\eta\in[0,\eta^*]}$ exhibits the* ***rotational instability*** *near time 0.*

Rotational instability

We can write

$$u^{(\eta)}(t,x) = \frac{e^{im(\theta+\gamma^{(\eta)}(t))}}{\sqrt{t^2+\eta^2}} Q^{(\eta)}_{|t|}\Big(\frac{r}{\sqrt{t^2+\eta^2}}\Big) + O_{H^1_m}(\alpha^*),$$

where $\gamma^{(\eta)}(t)$ satisfies

$$\begin{aligned} |\gamma^{(0)}(-\tau)| &\lesssim \alpha^*\tau, \\ \limsup_{\eta\to 0^+}\Big|\gamma^{(\eta)}(\tau)-\gamma^{(\eta)}(-\tau)-\Big(\frac{m+1}{m}\Big)\pi\Big| &\lesssim \alpha^*\tau, \end{aligned} \quad \text{for all small } \tau>0.$$

- When $\eta > 0$, one almost has

$$\gamma^{(\eta)}(t) \approx \frac{m+1}{m}\tan^{-1}\Big(\frac{t}{\eta}\Big)$$

  so the **abrupt spatial rotation** takes place on the time interval $|t| \lesssim \eta$.
- Notice that $u^{(0)} = u$ **does not** rotate at all.

Results in (NLS)

Our main result is analogous to mass-critical NLS, which are originally due to **Bourgain-Wang** ('97), and **Merle-Raphaël-Szeftel** ('13).
The *mass-critical nonlinear Schrödinger equation* on $\mathbb{R}^2$:

$$i\partial_t \psi + \Delta\psi + |\psi|^2\psi = 0, \tag{NLS}$$

where $\psi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$. There is a standing wave solution (but **not static**)

$$e^{it}R(x),$$

where R is a minimizer of $\frac{1}{2}\int|\psi|^2 + \frac{1}{2}\int|\nabla\psi|^2 - \frac{1}{4}\int|\psi|^4 = \frac{1}{2}M(\psi) + E_{NLS}(\psi)$, or R is the **ground state** soliton

$$\Delta R - R + R^3 = 0.$$

Applying the pseudoconformal symmetry,

$$S_{\mathrm{NLS}}(t,x) := \frac{1}{|t|}R\left(\frac{x}{|t|}\right)e^{\frac{i}{|t|}}e^{-i\frac{|x|^2}{4|t|}}, \qquad \forall t<0.$$

Bourgain-Wang solutions(NLS)

Theorem (Bourgain-Wang '97)

Let $\zeta^* : \mathbb{R}^2 \to \mathbb{C}$ be a profile that ***degenerates at the origin*** *at large order and is in some weighted Sobolev space. Then, there exists a (conditionally unique) solution ψ_{BW} to (NLS) such that*

$$\psi_{\mathrm{BW}}(t) - S_{\mathrm{NLS}}(t) \to \zeta^* \qquad \text{as } t \to 0.$$

Idea of proof

Via the pseudoconformal transform $\mathscr{C}$, it suffices to construct a solution $\mathscr{C}\psi$ such that $e^{-it}\mathscr{C}\psi(t) - R - e^{-it}\mathscr{C}\zeta(t) \to 0$ as $t \to \infty$, where $\zeta(t)$ is a solution to (NLS) with initial data ζ^*. Write the Duhamel formulation for $e^{-it}\mathscr{C}\psi(t)$ (from $t = +\infty$ to the present time) and run a contraction principle by exploiting the **decoupling**

$$|R(x)\mathscr{C}\zeta(t,x)| \lesssim t^{-A}, \qquad A \gg 1.$$

- Working directly with the pseudoconformal transform requires solutions to belong to a weighted Sobolev space. In case of (CSS), Q has polynomial tail $r^{-(m+2)}$ as $r \to \infty$. This **does not** belong to $H^{k,k}$ with k large.
- (Discussed later), there is a nontrivial **long-range interaction** in (CSS), induced from the gauge potential.

Instability of Bourgain-Wang solutions(NLS)

Pseudoconformal blow-up solutions are believed to be non-generic. Here is an instability result by **Merle-Raphaël-Szeftel**.

Theorem (Merle-Raphaël-Szeftel '13)

There is a continuous family of solutions ψ_η to (NLS) for $\eta \in [-1,1]$ such that

1. $(\eta = 0)\, \psi_0 = \psi_{\mathrm{BW}}$ *is the Bourgain-Wang solution,*
2. $(\eta > 0)\, \psi_\eta$ *scatters both forward and backward in time,*
3. $(\eta < 0)\, \psi_\eta$ *scatters backward and blows up forward in finite time under the log-log law, i.e.*

$$\|\nabla \psi_\eta(t)\|_{L^2} \approx c_* \left(\frac{\log|\log(T-t)|}{T-t} \right)^{\frac{1}{2}}.$$

- **No explicit use** of the pseudoconformal transform. Instead, they use modulation analysis with modified profiles, say $R_{\eta,b}$. Here, η is fixed and b is a parameter for the pseudoconformal phase $e^{-ib\frac{|y|^2}{4}}$.
- Instability direction is induced by ρ_{NLS}, which lies in the generalized null space of the linearized operator for (NLS).
- The case $\eta < 0$ falls into the negative energy and hence to the regime of stable log-log blow-up by Merle-Raphaël's works.

Comparison with (CSS) and (NLS)

- All the symmetries of (CSS) are valid for (NLS), including L^2-scaling and pseudoconformal symmetries. Conservation laws are also valid.
- Profiles Q and R:

$$-\left(\partial_{rr}+\frac{1}{r}\partial_r\right)Q+\left(\frac{m+A_\theta[Q]}{r}\right)^2 Q=Q^3-A_0 Q,$$
$$-\Delta R+R=R^3,$$

  Because of the mass-term, R shows exponential decay, whereas Q shows polynomial decay $r^{-(m+2)}$.

- **Generalized null spaces** of $\mathscr{L}_{\mathrm{NLS}}$ and $\mathscr{L}_Q$:

$$\begin{cases} i\mathscr{L}_{\mathrm{NLS}}\rho_{\mathrm{NLS}}=i|y|^2R, \\ i\mathscr{L}_{\mathrm{NLS}}i|y|^2R=4\Lambda R, \\ i\mathscr{L}_{\mathrm{NLS}}\Lambda R=-2iR, \\ i\mathscr{L}_{\mathrm{NLS}}iR=0, \end{cases} \qquad \begin{cases} i\mathscr{L}_Q\rho=iQ, \quad i\mathscr{L}_Q ir^2Q=4\Lambda Q, \\ i\mathscr{L}_Q iQ=0, \qquad i\mathscr{L}_Q\Lambda Q=0. \end{cases}$$

  Note that $i\mathscr{L}_{\mathrm{NLS}}\Lambda R\neq 0$ but $i\mathscr{L}_Q\Lambda Q=0$. This is again because $e^{it}R(x)$ is not a static solution to (NLS), but Q is a static solution to (CSS).

- The **self-duality** appears at the linearized level as

$$i\mathscr{L}_Q=iL_Q^*L_Q.$$

Comments on main theorems

- **Assumption (H).**
  1. **Degeneracy** of z^* at the origin $|z^*(r)| \lesssim \alpha^* r^{m+2}$. Required for decoupling estimates for the marginal interaction between $S(t)$ and z^*.
  2. **Long-range interaction**. After approximating $|S(t)|^2$ as a point charge at the origin, due to

$$m + A_\theta[S(t)] \approx m - 2(m+1) = -(m+2),$$

  the natural evolution equation for z is the $-(m+2)$-equivariant (CSS).
- **Assumption** $m \geq 1$ is required at many places.
  1. $S(t,r)$ is a H^1_m-solution if and only if $m \geq 1$.
  2. Nice embedding properties: $\dot{H}^1_m \hookrightarrow L^\infty$ and Hardy's inequality.
  3. Many other places where the proof breaks.
- **Interaction of $S(t)$ and z^*.** In contrast to (NLS), we have to incorporate the long-range (nonlocal) interaction between $S(t)$ and z. Thus,
  1. we evolve z under $-(m+2)$-equivariant (CSS),
  2. there is a phase correction $\gamma_{\mathrm{cor}}(t)$ in the theorem,
  3. but this does not change the blow-up rate.

Comments on main theorems

- **Rotational Instability**.
  1. The source of the instability is the phase rotation, which shows a sharp contrast to (NLS). Mathematically, the difference comes from that of the spectral properties of $\mathscr{L}_{\mathrm{NLS}}$ and $\mathscr{L}_Q$.
  2. When $\eta = 0$, $u^{(0)}$ does not rotate at all. But $u^{(\eta)}$ with $0 < \eta \ll 1$ shows a spatial rotation on $|t| \lesssim \eta$ by the angle
  $$\left(\frac{m+1}{m}\right)\pi.$$
  3. A rotational instability is observed in the energy-critical Schrödinger map (1-equivariant) by **Merle-Raphaël-Rodnianski '12**.

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Modulation analysis

We write

$$u^{(\eta)}(t,r) = \frac{e^{i\gamma(t)}}{\lambda(t)}[Q^{(\eta)}_{b(t)} + \varepsilon]\Big(t, \frac{r}{\lambda(t)}\Big) + z(t,r).$$

- $Q^{(\eta)}$ is some profile exhibiting the rotational instability with $Q^{(0)} = Q$.
- Pseudoconformal phase $f_b(r) = f(r)e^{-i\frac{b}{4}r^2}$.
- For given z^*, we fix evolution of $z(t,r)$ by (zCSS) equation (a small scattering global solution). (zCSS) is motivated to absorb the strong interaction between $S(t)$ and z.
- we have freedom to choose 3 conditions to fix dynamics of $b(t), \lambda(t), \gamma(t)$ and hence $\varepsilon(t,x)$.
- Initial dataa at $t = 0$

$$(\lambda(0), \gamma(0), b(0)) = (\eta, 0, 0), \qquad u^{(\eta)}(0,x) = \frac{1}{\eta}Q^{(\eta)}\Big(\frac{r}{\eta}\Big)e^{im\theta} + z^*(x).$$

- Establish uniform estimate (wrt η) for $\varepsilon, \lambda, \gamma, b$ by **bootstrapping** argument via **Laypunov method**.

$$b(t) \approx |t|, \quad \lambda(t) \approx \sqrt{t^2+\eta^2}, \quad \gamma(t) \approx \gamma_{\mathrm{cor}}(t) + (m+1)\tan^{-1}(\frac{t}{\eta}), \quad \text{and}$$

$$\lambda^{\frac{3}{4}}\|\varepsilon\|_{L^2} + \|\varepsilon\|_{\dot{H}^1_m} \lesssim \alpha^* \lambda^{m+2} + \lambda^{\frac{5}{4}}\eta^{\frac{3}{4}}.$$

- The blow-up solution is constructed by limiting $\eta \to 0$.

Pseudoconformal phase $Q_b(r) = Q(r)e^{-i\frac{b}{4}r^2}$

Recall:

$$\mathbf{D}_+^{(Q)} Q = 0 \quad \text{and} \quad f_b(y) := f(y)e^{-ib\frac{|y|^2}{4}}.$$

For a profile $Q^{(\eta)}$, assume $Q_{b(t)}^{(\eta)\sharp}$ solves (CSS). Then, by dynamic rescaling

$$\begin{aligned}
0 &= i\partial_t Q_b^{(\eta)\sharp} - L^*_{Q_b^{(\eta)\sharp}} \mathbf{D}_+^{(Q_b^{(\eta)\sharp})} Q_b^{(\eta)\sharp} \\
&= \frac{1}{\lambda^2}\Big[i\partial_s Q_b^{(\eta)} - i\frac{\lambda_s}{\lambda}\Lambda Q_b^{(\eta)} - \gamma_s Q_b^{(\eta)} - L^*_{Q_b^{(\eta)}} \mathbf{D}_+^{(Q_b^{(\eta)})} Q_b^{(\eta)} \Big]^\sharp \\
&= -\frac{1}{\lambda^2}\Big[(L^*_{Q^{(\eta)}} \mathbf{D}_+^{(Q^{(\eta)})} Q^{(\eta)})_b + i\Big(\frac{\lambda_s}{\lambda} + b\Big)\Lambda Q_b^{(\eta)} + \gamma_s Q_b^{(\eta)} - (b_s + b^2)\frac{|y|^2}{4} Q_b^{(\eta)} \Big]^\sharp.
\end{aligned}$$

where $\Lambda = 1 + r\partial_r$ is the L^2 scaling generator. When $\eta = 0$, the above computation suggests

$$\frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s = 0, \quad b_s + b^2 = 0.$$

This is satisfied by $S(t)$, i.e. $(b, \lambda, \gamma)(t) = (|t|, |t|, 0)$.

Dynamic rescaling

- Originally, we work with $u(t,x), z(t,x)$ but $Q_b^{(\eta)}(s,y), \varepsilon(s,y)$ where $y = \frac{x}{\lambda}$.
- $\sharp$ **and** $\flat$ **notations**. Let λ and γ be given. For a function $f(y)$, we convert f to a function on x as

$$f^\sharp(x) := \frac{1}{\lambda} f\left(\frac{x}{\lambda}\right) e^{i\gamma}.$$

Similarly, we convert a function $g(x)$ to a function on y as

$$g^\flat(y) := \lambda g(\lambda y) e^{-i\gamma}.$$

- **Dynamic rescaling**. We introduce (s,y) variables as

$$\frac{ds}{dt} = \frac{1}{\lambda^2(t)}; \qquad y := \frac{x}{\lambda(t)}.$$

Then,

$$\partial_t f^\sharp = \frac{1}{\lambda^2}\left[\partial_s f - \frac{\lambda_s}{\lambda}\Lambda f + i\gamma_s f\right]^\sharp,$$

$$\partial_s g^\flat = \lambda^2\left[\partial_t g + \frac{\lambda_t}{\lambda}\Lambda g - i\gamma_t g\right]^\flat.$$

- In this notation, the ansatz is

$$u(t,x) = (Q_b^{(\eta)} + \varepsilon)^\sharp + z, \quad \text{or } u^\flat(s,y) = (Q_b^{(\eta)} + \varepsilon) + z^\flat$$

Profile $Q^{(\eta)}$

Our profile $Q^{(\eta)}$ will be obtained by perturbing the formal parameter ODEs

$$\frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s = 0, \quad b_s + b^2 = 0.$$

- **(NLS) case**: Merle-Raphaël-Szeftel introduced the η-parameter **only** in $b_s + b^2 = -\eta$. This is **forbidden** in (CSS), due to the spectral property of $\mathcal{L}_Q$.
- **Crucial observation:** If we introduce η to the phase rotation instead, a formal computation based on the Pohozaev identity yields that $b_s + b^2$ must have a nontrivial $O(\eta^2)$-term:

$$\left[\frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s = \eta\right] \quad \Longrightarrow \quad b_s + b^2 \approx -c\eta^2, \quad c > 0.$$

Solving this ODE system, one obtains a **rotational instability**.

Profile $Q^{(\eta)}$

Substituting the formal parameter law, we should solve

$$L^*_{Q^{(\eta)}}\mathbf{D}_+^{(Q^{(\eta)})}Q^{(\eta)} + \eta Q_b^{(\eta)} + c\eta^2 \tfrac{|y|^2}{4} Q_b^{(\eta)} = 0. \qquad (1)$$

This is a second-order **nonlocal** PDE.

- **Difficulty for the construction**. It is customary to Taylor expand $Q^{(\eta)}$ in the η-variable, which **loses** r^2 decay at each step. This is especially dangerous when m is small. Moreover, as $Q^{(\eta)}$ is expected to have an exponential decay, the η-expansion will require a truncation and complicate the argument.
- **Nonlinear ansatz**: it turns out that we can use **self-duality** to reduce (1) to a first-order differential equation.

$$\begin{cases} \mathbf{D}_+^{(Q^{(\eta)})}P^{(\eta)} = 0, \\ Q^{(\eta)} = e^{-\eta\frac{r^2}{4}}P^{(\eta)} \end{cases} \implies \begin{cases} L^*_{Q^{(\eta)}}\mathbf{D}_+^{(Q^{(\eta)})}Q^{(\eta)} + \eta\theta_\eta Q_b^{(\eta)} + \eta^2\frac{|y|^2}{4}Q_b^{(\eta)} = 0, \\ \theta_\eta = \frac{1}{2}\int |Q^{(\eta)}|^2 r' dr' - (m+1) \approx m+1. \end{cases}$$

- **Formal parameter law for** $Q^{(\eta)}$:

$$\frac{\lambda_s}{\lambda} + b = 0, \quad \gamma_s = \eta\theta_\eta, \quad b_s + b^2 + \eta^2 = 0.$$

Hence,

$$\lambda(t) = \sqrt{t^2+\eta^2}, \quad \gamma(t) = \theta_\eta \tan^{-1}\frac{t}{\eta}, \quad b(t) = -t.$$

Interaction between $Q_b^{(\eta)\sharp}$ and z

- **Effect** $Q_b^{(\eta)\sharp} \to z$: There is a long-range interaction. A typical one is

$$\Big(\frac{m + A_\theta[Q_b^{(\eta)\sharp} + z]}{r}\Big)^2 z \approx \Big(\frac{m + A_\theta[Q_b^{(\eta)\sharp}] + A_\theta[z]}{r}\Big)^2 \approx \Big(\frac{-(m+2)}{r}\Big)^2 z.$$

  Thus z evolves under $-(m+2)$-equivariant (CSS) =:(zCSS).

- **Effect** $z \to Q_b^{(\eta)\sharp}$: Correction in the phase.

$$\theta_{z \to Q_b^{(\eta)\sharp}} Q_b^{(\eta)}$$

  that leads to the phase correction

$$\gamma_{\mathrm{cor}}^{(\eta)}(t) := -\int_0^t \theta_{z \to Q_b^{(\eta)\sharp}} dt'.$$

- **Case of (NLS)**: the nonlinearity $|\psi|^2\psi$ is local. Thus the interaction between R_b and $\zeta^\flat$ becomes small due to **fast decay** of R_b and **degeneracy** of $\zeta^\flat$ at the origin. Thus it suffices to evolve ζ under (NLS) itself, without any forcing term.

Evolution of ε

Now the equation for ε becomes

$$
\begin{aligned}
& i\partial_s \varepsilon - \mathscr{L}_{w^\flat}\varepsilon + ib\Lambda\varepsilon - \eta\theta_\eta \varepsilon \\
&= i\Big(\frac{\lambda_s}{\lambda} + b\Big)\Lambda(Q_b^{(\eta)} + \varepsilon) + (\widetilde{\gamma}_s - \eta\theta_\eta)Q_b^{(\eta)} + (\gamma_s - \eta\theta_\eta)\varepsilon \\
&\quad - (b_s + b^2 + \eta^2)\tfrac{|y|^2}{4} Q_b^{(\eta)} + \widetilde{R}_{Q_b^{(\eta)}, z^\flat} + V_{Q_b^{(\eta)} - Q_b} z^\flat + R_{u^\flat - w^\flat}.
\end{aligned}
$$

- Here, $w := Q_b^{(\eta)} + z^\flat$ and $\widetilde{\gamma}_s := \gamma_s + \theta_{z^\flat \to Q_b^{(\eta)}}$.
- The effect from $Q_b^{(\eta)}$ to $z$is removed by z-evolution.
- $\widetilde{R}_{Q_b^{(\eta)}, z^\flat}$ is the marginal interaction satisfying $\|\widetilde{R}_{Q_b^{(\eta)}, z^\flat}\|_{H^1} \lesssim \alpha^* \lambda^{m+3} |\log \lambda|$.
- $R_{u^\flat - w^\flat} = O(\varepsilon^2)$.
- $V_{Q_b^{(\eta)} - Q_b}$ arises from the difference of $Q_b^{(\eta)}$ and Q_b.

Choice of modulation parameters

We haven't specified the choice of b, λ, γ. We spend three degrees of freedom by

- two (generic) orthogonality conditions $\Rightarrow$ **Coercivity** $(\varepsilon, \mathscr{L}_Q \varepsilon) \gtrsim \|\varepsilon\|^2_{\dot{H}^1_m}$,
- one **dynamical law** $\Rightarrow 2(\frac{\lambda_s}{\lambda} + b) - (b_s + b^2 + \eta^2) = 0$. We are motivated to this choice to delete terms having dangerous spatial decay:

$$
\begin{aligned}
& i\Big(\frac{\lambda_s}{\lambda} + b\Big)\Lambda Q_b^{(\eta)} - (b_s + b^2 + \eta^2)\frac{|y|^2}{4} Q_b^{(\eta)} \\
&= i\Big(\frac{\lambda_s}{\lambda} + b\Big)[\Lambda Q^{(\eta)}]_b + \underbrace{\Big[2\Big(\frac{\lambda_s}{\lambda} + b\Big) - (b_s + b^2 + \eta^2)\Big]}_{=0}\frac{|y|^2}{4} Q_b^{(\eta)}.
\end{aligned}
$$

- The ε-equation is now simplified:

$$
\begin{aligned}
& i\partial_s \varepsilon - \mathscr{L}_{w^\flat}\varepsilon + ib\Lambda\varepsilon - \eta\theta_\eta\varepsilon \\
&= i\Big(\frac{\lambda_s}{\lambda} + b\Big)([\Lambda Q^{(\eta)}]_b + \Lambda\varepsilon) + (\widetilde{\gamma}_s - \eta\theta_\eta)Q_b + (\gamma_s - \eta\theta_\eta)\varepsilon \\
&\quad + \widetilde{R}_{Q_b^{(\eta)}, z^\flat} + V_{Q_b^{(\eta)} - Q_b} z^\flat + R_{u^\flat - w^\flat}.
\end{aligned}
$$

Lyapunov/virial Functional

In order to close the bootstrap, we should be able to estimate $\|\varepsilon\|_{\dot{H}^1_m}$ and $\|\varepsilon\|_{L^2}$ by propagating smallness of ε at $(\varepsilon(0)=0)$ to the past times. For this, we use a Lyapunov method. **Martel** ('05 AJM) was the first to use energy method in backward construction.

- In view of coercivity it is natural to start with the energy functional. However, it does not suffice and we need to add a correction. The correction term is motivated from the observation that ε indeed evolves under

$$i\partial_s\varepsilon - \mathscr{L}_{w^\flat}\varepsilon + ib\Lambda\varepsilon - \eta\theta_\eta\varepsilon \approx 0.$$

  The energy functional is only adapted to $i\partial_s\varepsilon - \mathscr{L}_{w^\flat}\varepsilon \approx 0$.

- Moreover, we also need an averaging argument. As a result, we use

$$\mathscr{I} := \lambda^{-2}\Big(E^{(\mathrm{qd})}_{w^\flat}[\varepsilon] + \frac{\eta\theta_\eta}{2}M[\varepsilon] + \frac{2b}{\log A}\int_{A^{1/2}}^{A}\Phi_{A'}[\varepsilon]\frac{dA'}{A'}\Big).$$

  - Here, $E^{(\mathrm{qd})}_{w^\flat}[\varepsilon] := E[w^\flat+\varepsilon] - E[w^\flat] - (\frac{\delta E}{\delta u}\Big|_{u=w^\flat}, \varepsilon)_r$,
  - $\Phi_A[\varepsilon]$ is a localized virial functional. The localized virial correction $b\Phi_A[\varepsilon]$ was first introduced by **Raphaël** and **Szeftel** ('11 JAMS).

Final comments

- **Long-range interaction between $Q_b^{(\eta)\sharp}$ and** z requires two corrections: the evolution of $z(t,x)$ and phase correction of $Q_b^{(\eta)}$.
- **New instability mechanism**: $\frac{m+1}{m}\pi-$angle spartial rotation near blow-up time.
- **Self-Duality** plays a crucial role in several places: Informations on linearized operator, construction of modified profile $Q^{(\eta)}$
- The prescribed asymptotic profile z^* require **one additional condition (H).** (cf. Krieger-Schlag 10' 1D NLS)
- There should be a separate argument of L^2 control, as the coercivity only control $\dot{H}^1$.

Thanks for your attention!