Large deviations for conservative, stochastic PDE and non-equilibrium fluctuations

Benjamin Gess
Max Planck Institute for Mathematics in the Sciences, Leipzig
& Universität Bielefeld

Asia-Pacific Analysis and PDE Seminar
December 2020

joint work with: Ben Fehrman, Nicolas Dirr.
[Fehrman, G.; arxiv, 2020], [Dirr, Fehrman, G.; arxiv, 2020].

1. Conservative SPDE as fluctuating continuum models

2. Two ways to the LDP, the skeleton equation

The zero range process

(could also consider simple exclusion, independent particles).

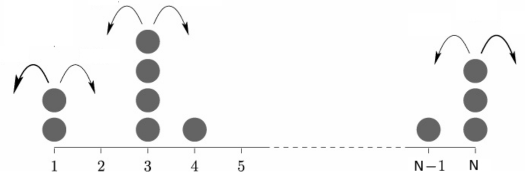

Figure: Harris, Rákos, Schütz; 2005

- State space $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N}$, i.e. configurations $\eta : \mathbb{T}_N \to \mathbb{N}_0$: System in state η if container k contains $\eta(k)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \to \mathbb{R}_0^+$.
- Translation invariant, asymmetric, zero mean transition probability

$$p(k,l) = p(k-l), \quad \sum_k kp(k) = 0.$$

- Markov jump process $\eta(t)$ on $\mathbb{M}_N$.
- $\eta(k,t) =$ number of particles in box k at time t.

- Hydrodynamic limit? Multi-scale dynamics

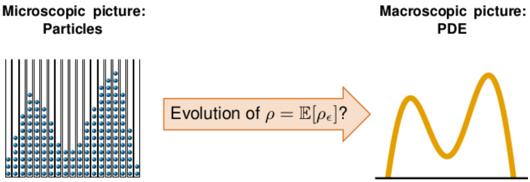

Figure: see Zimmer et. al.

- Empirical density field

$$\mu^N(x,t) := \frac{1}{N}\sum_k \delta_{\frac{k}{N}}(x)\eta(k, tN^2).$$

- [Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\, dx$$

with

$$\partial_t \bar{\rho} = \frac{1}{2}\partial_{xx}\Phi(\bar{\rho})$$

with Φ the mean local jump rate $\Phi(\rho) = \mathbb{E}_{\nu_\rho}[g(\eta(0))]$.

Rate of convergence?

- [Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988]: Fluctuation density fields

$$
\begin{aligned}
Y^N(x,t) &= \frac{1}{\sqrt{N}} \sum_k \delta_{\frac{k}{N}}(x)[\eta(k,tN^2) - \mathbb{E}\eta(k,tN^2)] \qquad (\star) \\
&= \sqrt{N}(\mu^N(x,t) - \mathbb{E}\mu^N(x,t))
\end{aligned}
$$

  for $t \geq 0$. Then,

$$
\mathscr{L}(Y^N) \rightharpoonup^* \mathscr{L}(Y) \text{ for } N \to \infty
$$

  with Y the solution to

$$
dY(x,t) = \partial_{xx}(\Phi'(\bar{\rho}(x,t))Y(x,t))\,dt + \partial_x(\sqrt{\Phi(\bar{\rho}(x,t))}dW(t))
$$

  with dW space-time white noise.

- Therefore, expect

$$
d(\mu^N, \bar{\rho}\,dx) \approx N^{-\frac{1}{2}}.
$$

- Re-interpret $(\star)$ as fluctuation correction

$$\begin{aligned}\mu^N(x,t) &= \sqrt{N} Y^N(x,t) + \mathbb{E}\mu^N(x,t) \\ &= \underbrace{\frac{1}{\sqrt{N}} Y^N(x,t) + \bar{\rho}(x,t)}_{:=\bar{\rho}^N(x,t)} + \underbrace{\mathbb{E}\mu^N(x,t) - \bar{\rho}(x,t)}_{=O(N^{-1})}.\end{aligned}$$

- Hence,

$$d(\mu^N, \bar{\rho}^N) \approx N^{-1}.$$

and notice that the *linearly* corrected continuum model $\bar{\rho}^N(x,t)$ satisfies

$$d\bar{\rho}^N(x,t) = \partial_{xx}(\Phi'(\bar{\rho}(x,t))\bar{\rho}^N(x,t))\,dt + \frac{1}{\sqrt{N}}\partial_x(\sqrt{\Phi(\bar{\rho}(x,t))}dW(t)) \quad (\star)$$

i.e. a linear stochastic PDE with noise of small amplitude.

- Rare events? For $(\star)$ we have rare events

$$\mathbb{P}[\bar{\rho}^N \approx \rho\, dx] \approx \exp\{-N\, \bar{I}_0(\rho\, dx)\},$$

with

$$\bar{I}_0(\rho\, dx) = \inf\left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dx ds : g \in L^2_{t,x},\, \partial_t \rho = \partial_{xx}(\Phi'(\bar{\rho})\rho) + \partial_x(\Phi^{\frac{1}{2}}(\bar{\rho})g) \right\}.$$

- [Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995]: Let now ρ_0 constant. Then, informally,

$$\mathbb{P}[\mu^N \approx \rho\, dx] \approx \exp\{-N\, I_0(\rho\, dx)\},$$

with rate function

$$I_0(\rho\, dx) = \inf\left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dxds : g \in L^2_{t,x}, \underbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho) g)}_{\text{"skeleton equation"}} \right\}.$$

- Note: This does **not** coincide with the rate function of the linearly corrected continuum model $\bar{\rho}^N$,

$$\bar{I}_0(\rho\, dx) = \inf\left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dxds : g \in L^2_{t,x}, \partial_t \rho = \partial_{xx}(\Phi'(\bar{\rho})\rho) + \partial_x(\Phi^{\frac{1}{2}}(\bar{\rho}) g) \right\}.$$

- Ansatz: Derive a **nonlinear** fluctuating continuum model to simultaneously obtain higher order approximation and correct rare event behavior.

Ansatz: Langevin dynamics

$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^N)}dW_t\right). \qquad (\star)$$

Model case: Dean-Kawasaki, independent particles, $\Phi(\rho) = \rho$, i.e.

$$\partial_t \rho = \partial_{xx}\rho + \frac{1}{\sqrt{N}}\partial_x(\sqrt{\rho}dW_t).$$

Informal justification:

1. Physics: Fluctuation-dissipation relation, "fluctuating hydrodynamics"
2. Mean behavior / law of large numbers

$$\rho^N \to \bar{\rho} \quad \text{as } N \to \infty.$$

3. Central limit fluctuations: $Y^N := \sqrt{N}(\rho^N - \bar{\rho})$. Then, $\mathscr{L}(Y^N) \rightharpoonup^* \mathscr{L}(Y)$ with

$$\partial_t Y = \partial_{xx}\left(\Phi'(\bar{\rho})Y\right) + \partial_x\left(\sqrt{\Phi(\bar{\rho})}dW_t\right).$$

4. Large deviations: See below, large deviations of $(\star)$ are the same as for μ^N.

Informally, correct rare events:

- Informally applying the contraction principle to the solution map

$$F : \frac{1}{\sqrt{N}} dW \mapsto \rho$$

  yields as a rate function

$$I(\rho) = \inf\{I_{dW}(g) : F(g) = \rho\}.$$

- Schilder's theorem for Brownian sheet suggests

$$I_{dW}(g) = \int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt.$$

- Get

$$I(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt : \partial_t \rho = \partial_{xx}(\Phi(\rho)) + \partial_x\left(\sqrt{\Phi(\rho)} g\right)\right\}.$$

- Obstacle $$\partial_t \rho = \partial_{xx}(\Phi(\rho)) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho)}dW_t\right)$$
  1. not well-posed, supercritical -> no regularity structures
  2. Renormalization? Does renormalization appear in rate function? E.g. compare $\Phi^4_{2/3}$ [Hairer, Weber; 2014].
- Decorrelation length of discrete system $= \frac{1}{N}$.
- Ansatz: joint limit "small noise, ultraviolet cutoff"

$$\partial_t \rho^{N,K} = \partial_{xx}\left(\Phi(\rho^{N,K})\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^{N,K})}\circ dW^K_t\right)$$

where $W^K = \sum_{k=1}^K e_k \beta^k$ is a spectral (smooth) approximation of $W = \sum_{k=1}^\infty e_k \beta^k$.
- Gives the correct rate function for $\frac{1}{N} << \frac{1}{K}$.

Note: This is a particular case in which the link between *Macroscopic fluctuation theory* [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and *fluctuating hydrodynamics* [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.

Two ways to the LDP, the skeleton equation

1. Conservative SPDE as fluctuating continuum models
2. Two ways to the LDP, the skeleton equation

- In the following concentrate on the case
$$\Phi(\rho) = \rho^m, \quad m \geq 1.$$
- We consider stochastic PDE of the type
$$\partial_t \rho^{N,K} = \Delta\left((\rho^{N,K})^m\right) + \frac{1}{\sqrt{N}} \mathrm{div}\left((\rho^{N,K})^{\frac{m}{2}} \circ dW_t^K\right), \qquad (*)$$
on $\mathbb{T}^d \times (0,\infty)$, where $W^K = \sum_{k=1}^K e_k \beta^k$.
- Pathwise well-posedness of $(*)$: [Lions, Souganidis; 1998ff], [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2017], [Dareiotis, G.; 2019].

Two ways to the LDP:

1. Γ-convergence of the rate functional: $N \uparrow \infty$ yields LDP for $(*)$ with rate function
$$I^K(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}^d} |g|^2 \, dxdt : \partial_t \rho = \partial_{xx}\rho^m + \partial_x\left(\rho^{\frac{m}{2}} P^K g\right)\right\}.$$
Then consider $K \uparrow \infty$.
2. Joint scaling: Weak convergence approach to LDP ($\frac{1}{N} << \frac{1}{K}$).

- Both approaches crucially depend on understanding the skeleton PDE.
- The skeleton equation

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}\left(\rho^{\frac{m}{2}} g(t,x)\right) \qquad (*)$$
$$\rho(0,x) = \rho_0(x),$$

  with $g \in L^2_{t,x}$?
- This leads to the key problem

Problem

1. *Existence and uniqueness of solutions to* $(*)$.
2. *Stability of solutions: Let* $g^n \rightharpoonup g$ *in* $L^2_{t,x}$ *with corresponding solutions* ρ^n, ρ. *Then*
$$\rho^n \to \rho$$
   in $L^\infty_t L^1_x$.

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Scaling and criticality of the skeleton equation

- We consider
$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$
with $g \in L^q(\mathbb{R}_{+,t}; L^p(\mathbb{R}^d_x; \mathbb{R}^d_x))$ and $\rho_0 \in L^r(\mathbb{R}^d_x)$.
- Via rescaling (“zooming in”):
    - $p = q = 2$ is critical.
    - $r = 1$ is critical, $r > 1$ is supercritical.

Apriori-bounds and energy space

- Consider
$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \qquad (*)$$
with $g \in L^2(\mathbb{R}_{+,t}; L^2(\mathbb{R}^d_x; \mathbb{R}^d_x))$.
- L^1 estimate only gives
$$\int_{\mathbb{T}^d} \rho(t,x)dx = \int_{\mathbb{T}^d} \rho_0(x)dx.$$
- Use entropy-entropy dissipation: Evolution of entropy given by $\int_{\mathbb{T}^d} \log(\rho)\rho$. Informally gives
$$\int_{\mathbb{T}^d} \log(\rho)\rho \, dx\Big|_0^t + \int_0^t \int_{\mathbb{T}^d} (\nabla \rho^{\frac{m}{2}})^2 \lesssim \int_0^t \int_{\mathbb{T}^d} g^2.$$
- Caution: Can only be true for non-negative solutions.
- Non-standard weak solutions, rewriting $(*)$ as
$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d$$
- Conclusion: Have to prove uniqueness within this class of solutions.

Ansatz for uniqueness: Show that every weak solution is a renormalized entropy solution (extending the concepts of DiPerna-Lions, Ambrosio to nonlinear PDE).

Theorem

A function $\rho \in L_t^\infty L_x^1$ is a weak solution to

$$\partial_t \rho = 2div(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + div(\rho^{\frac{m}{2}} g)$$

if and only if ρ is a renormalized entropy solution.

Uniqueness for renormalized entropy solutions (variable doubling)

- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

$$q(x,\xi,t) = \delta(\xi - \rho(x,t)) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2$$

  does not satisfy

$$\lim_{|\xi|\to\infty} \int_{t,x} q(x,\xi,t)\, dxdt = 0.$$

- Established arguments [Chen, Perthame; 2003] not applicable.

Theorem (The skeleton equation)

Let $g \in L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$, $\rho_0 \in L^1(\mathbb{T}^d)$ *non-negative and* $\int \rho_0 \log(\rho_0)dx < \infty$, $m \in [1,\infty)$.

1. *There is a unique weak solution*

$$\partial_t \rho = \Delta \rho^m + div(\rho^{\frac{m}{2}} g) \quad on\ \mathbb{R}_+ \times \mathbb{T}^d. \tag{*}$$

   For two weak solutions $\rho^1, \rho^2 \in L^\infty([0,T];L^1(\mathbb{T}^1))$ *we have*

$$\|\rho^1-\rho^2\|_{L^\infty([0,T];L^1(\mathbb{T}^d))} \le \|\rho_0^1-\rho_0^2\|_{L^1(\mathbb{T}^d)}.$$

2. *Let* $\{g_n\}_{n\in\mathbb{N}} \subseteq L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$ *with*

$$\lim_{n\to\infty} g_n = g \ \ \textit{weakly in} \ \ L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$$

   and let $\rho_n \in L^1([0,T];L^1(\mathbb{T}^d))$ *be the corresponding solutions with control* g_n*. Then,*

$$\lim_{n\to\infty} \rho_n = \rho \ \ \textit{strongly in} \ \ L^1([0,T];L^1(\mathbb{T}^d))$$

   where $\rho \in L^1([0,T];L^1(\mathbb{T}^d))$ *is the solution with control* g*.*

Consider

$$d\rho^N = \Delta(\rho^N)^m\,dt + \frac{1}{\sqrt{N}}\mathrm{div}\left(\Phi^{\frac{1}{2}}_{n(N)}(\rho^N)\circ dW^{K(N)}(t)\right).$$

Theorem (Large deviation principle)

Let $K(N), n(N) \to \infty$ with $\frac{K(N)^3}{N} \to 0$ for $N \to \infty$. For $\rho_0 \in L^{m+1}(\mathbb{T}^d)$ and $\rho \in L^\infty([0,T];L^1(\mathbb{T}^d))$ let

$$I_{\rho_0}(\rho) := \inf\left\{\frac{1}{2}\int_0^T \|g(s)\|^2_{L^2_x}ds : \; g \in L^2_{t,x},\, \partial_t\rho = \Delta\rho^m + div(\rho^{\frac{m}{2}}g)\right\}.$$

Then, the family $\{\rho^N\}$ satisfies the large deviation principle on $L^\infty([0,T];L^1(\mathbb{T}^d))$ with good rate function I_{ρ_0}, uniformly on compact subsets of $L^{m+1}(\mathbb{T}^d)$.

- K. Dareiotis and B. Gess.
  Nonlinear diffusion equations with nonlinear gradient noise.
  Electronic Journal of Probability, 25:Paper No. 35, 43, 2020.
- N. Dirr, B. Fehrman, and B. Gess.
  Conservative stochastic PDE and fluctuations of the symmetric simple exclusion process.
  arXiv:2012.02126 [math], Dec. 2020.
- B. Fehrman and B. Gess.
  Well-posedness of nonlinear diffusion equations with nonlinear, conservative noise.
  Archive for Rational Mechanics and Analysis, 233(1):249–322, 2019.
- B. Fehrman and B. Gess.
  Large deviations for conservative stochastic PDE and non-equilibrium fluctuations.
  arXiv:1910.11860 [math], Mar. 2020.
- B. Gess and P. E. Souganidis.
  Scalar conservation laws with multiple rough fluxes.
  Commun. Math. Sci., 13(6):1569–1597, 2015.
- B. Gess and P. E. Souganidis.
  Stochastic non-isotropic degenerate parabolic–hyperbolic equations.
  Stochastic Process. Appl., 127(9):2961–3004, 2017.